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STATISTICAL RISK PROPERTIES OF THE LOGISTIC SUPPORT COST COMMIT--ETC(U)
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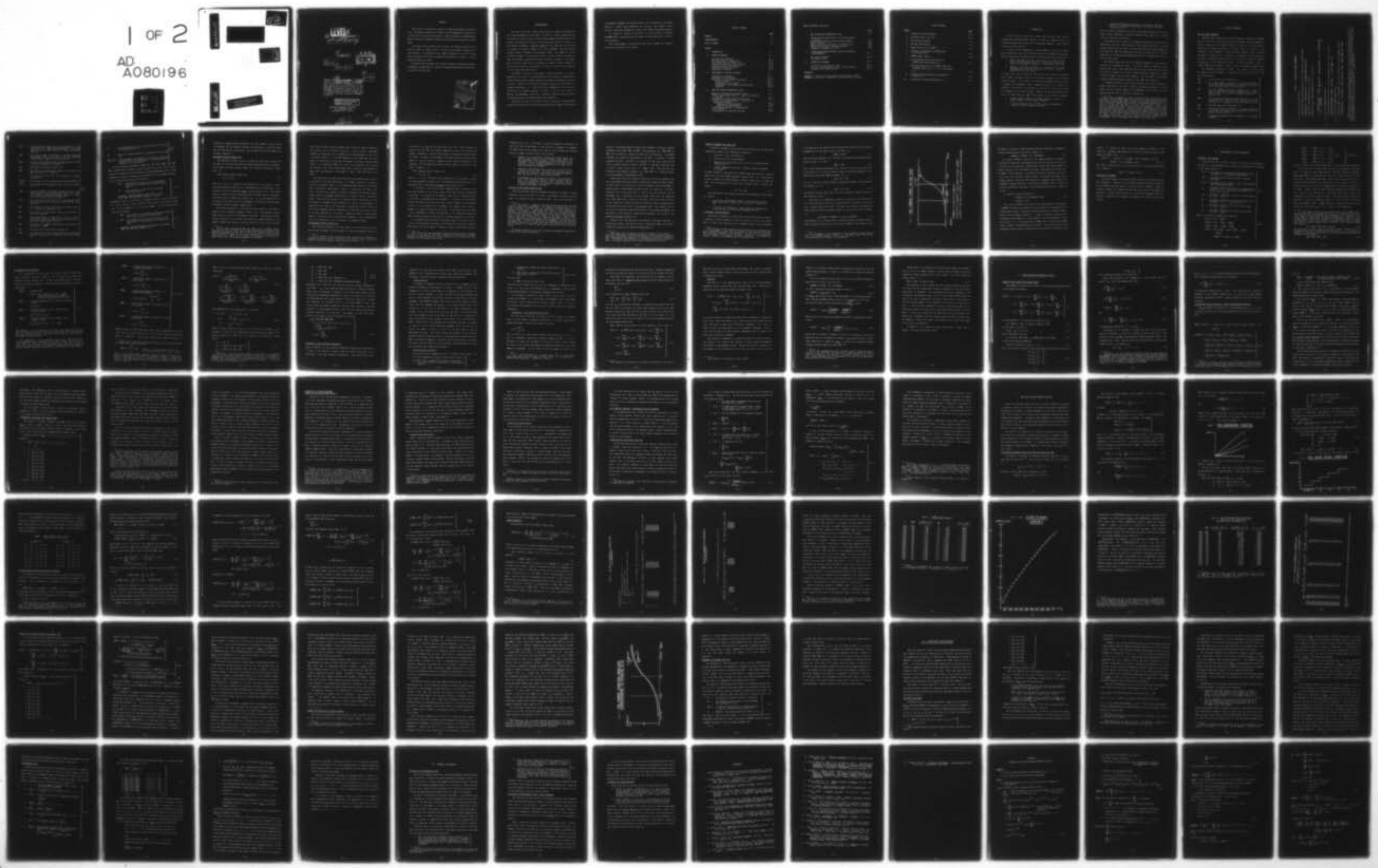

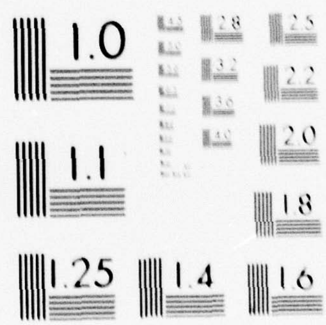
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STATISTICAL RISK PROPERTIES OF THE LOGISTIC SUPPORT COST COMMITMENT

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PREFACE

This report is addressed to analysts in Operations Research and Statistics. It presents the results of research into the mathematical properties of a broad class of statistical estimators. Those estimators are complex mathematical functions of simpler estimators whose statistical properties are well known.

For clarity and to indicate their utility, the research results are set forth in terms of a specific case. In that case the complex estimator is a cost function. It incorporates such simpler estimators as rates of occurrence, durations of activity, and physical distribution of activity. It also includes constant cost rates.

We believe that analogies can be drawn to numerous problems which appear in different settings and are stated in different nomenclature but which are, in abstract form, the same.

ACKNOWLEDGMENTS

The author would like to thank several people for support in carrying out the research documented in this report. Mr. Perry C. Stewart, Deputy Director of Acquisition Policy at Hq. Air Force Systems Command (formerly Chief of the Concepts and Analysis Directorate of the Acquisition Logistics Division, Air Force Logistics Command), originally suggested to the author the critical need for analysis of Logistic Support Cost (LSC) Commitment risks and provided informal support throughout the subsequent study. Maj. Samuel B. Graves, currently assigned to the Assistant Chief of Staff, Studies and Analysis, USAF, The Pentagon, performed preliminary simulation analyses of LSC Commitment statistical risks while in residence as a Master's degree candidate at the Air Force Institute of Technology. This work laid much of the foundation for the research described herein.

The author is grateful for the support of three consultants. Dr. John A. Muckstadt of the School of Operations Research and Industrial Engineering at Cornell University made numerous important contributions throughout the study. Dr. Alan J. Truelove of the School of Business, University of the District of Columbia, Washington, D. C., made very helpful suggestions in the areas of statistical and computer analysis. Dr. Saul I. Gass of the College of Business and Management, University of Maryland, evaluated interim results, and explored potentially helpful optimization techniques.

Within LMI, Dr. Marco R. Fiorello, leader of the project under which the research was carried out, provided continuing evaluation support and sustained

encouragement throughout the research effort. His contribution to the development of a proper study perspective was critical. Mrs. Jeanne M. White provided considerable mathematical analysis and computer programming support.

Mr. Russell R. Shorey of the Office of the Secretary of Defense, OASD(MRA&L)(WR), provided evaluative support for which the author is particularly grateful.

With acknowledgment of these persons the author assumes full responsibility for the content of this report.

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I. INTRODUCTION

Of major concern to Air Force leaders today are the rising costs of operating and supporting weapon systems. Two key cost elements are the cost of acquiring spare items and the cost of maintenance manpower. These range in billions of dollars per year.

It would be most advantageous to the Government to be able to identify and estimate these costs more effectively during the equipment design phase. Better targeting and tracking of these costs during this period would lead to:

a) better Government decisions regarding how the weapon system design, the operational scenario, or the logistics and maintenance policies might be modified to reduce these costs, and

b) improved contractual procedures for motivating the system development and production contractor to strive to produce a minimum life cycle cost design.

In recent years, the Air Force has used several new contractual procedures which attempt to estimate, target, and track logistic support costs during the acquisition phase. One of these is a contractual mechanism known as a Logistic Support Cost Commitment (LSCC), sometimes referred to as a Support Cost Guarantee. The objective of the LSCC is to motivate the contractor to design his equipment to have reduced logistic support costs through increased reliability and maintainability (R&M) when fielded. It has three key elements:

1. A target logistic support cost (TLSC), defined in terms of a logistic support cost (LSC) model framework.

2. A field verification test procedure, including computation of a "measured" logistic support cost (MLSC).

3. A positive and negative adjustment to the contract, which is a function of verification test results (particularly the MLSC) vis-a-vis the TLSC.[1]

The cost figures in question are verified in the sense that the failure rates, repair times, percentages of failures repaired at base, depot, etc., of which they are a function, are verified.

The purpose of this report is to document research into the statistical properties of the LSCC. The objectives of that research have been (1) to gain greater insight into the risks of drawing false conclusions from observed LSC data under the terms and conditions of the LSCC provision, and (2) to develop guidelines for structuring and managing the LSCC in a manner that takes account of these risks.

The reader should note that the research documented in the report is primarily of a mathematical nature. It does not address in depth any of the numerous qualitative issues regarding LSCC use, such as what test data should be collected, adequacy of test data collection methods, synthesis of interim contractor support into the LSCC framework, and the impact of government initiated engineering change proposals on LSCC legal enforceability.

A summary discussion of the research results can be found, along with a general discussion of the LSCC's incentive transmitting features, in an LMI report entitled "Logistic Support Cost Commitments for Life Cycle Cost Reduction" (LMI Task 75-1/5, June 1977 (reference 4)).

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[1] Two recent procurements in which the LSCC has been incorporated are the AN/ARN-101 tactical LORAN, managed by the Air Force Electronics Systems Division (ESD) at Hanscom AFB, Mass., and the F-16 aircraft, managed by the Air Force Aeronautical Systems Division (ASD) at Wright-Patterson AFB, Ohio. In the case of the F-16, the LSCC is in effect at two levels. First, there exists a "system-level" TLSC--a target LSC for the aggregate of 280 aircraft line replaceable units (LRUs). The LSCC includes a multi-million dollar award fee provision with respect to this target but no negative incentive. Second, there exists a TLSC specifically with respect to three of the aircraft's high cost LRUs: the fire control radar, the electro-optical (E-O) display, and the E-O display electronics. The LSCC includes provisions for both positive and negative contract adjustments with respect to the latter TLSC.

II. ANALYSIS FRAMEWORK

THE COST MODEL FRAMEWORK

The LSCC uses a simplified cost model framework to represent Government LSCs as a function of contractor-controllable equipment logistics parameters. Figure 1 reflects an LSC framework typical of those used in recent LSCC applications (all future discussion of research results is carried on with reference to this framework). Such a cost model framework (CMF) is usually developed by program office or acquisition management staff personnel and provided to competing contractors as part of the LSCC provisions, ultimately becoming part of the negotiated contract. The CMF establishes a basis for formal communications between a Contractor and the Government regarding LSCs.

The parameters appearing in Figure 1 are defined as follows (where G = supplied by the Government, C = supplied by the contractor, and M = computed by the model):

BLR	=	the base labor rate (\$/man-hour) (G)	II.1
BMH	=	the average number of man-hours to perform intermediate level (base shop) maintenance on a removed item including fault isolation, repair, and verification (C)	
BMR	=	the base consumable material consumption rate in \$/man-hour (including minor items of supply such as nuts, washers, rags, etc., which are consumed during repair of the item) (G)	
BRCT	=	the average base repair cycle time in months, i.e., the elapsed time for a RTS item from removal of the failed item until it is returned to base serviceable stock (G)	
DLR	=	the depot labor rate (\$/man-hour) (G)	
DMH	=	the average number of man-hours to perform depot-level maintenance on a removed item including fault isolation, repair, and verification (C)	
DMR	=	the depot consumable material consumption rate (analogous to BMR) (G)	

FIGURE 1. A TYPICAL LSC MODEL FRAMEWORK*

C_1 = cost of initial spare items

= (Cost of base repair pipeline spares) + (Cost of depot repair pipeline spares)

$= (NB)(\underline{STK})(UC) + [(PFFH)(UF)(QPA)(1 - \underline{RIP})(\underline{NRTS})/(\underline{MTBF})](DRCT)(UC)$

where STK is the minimum value of i such that

$$XBO(i,\theta t) = \sum_{x>i} (x-i)P(x \mid \theta t) \leq EBO, P(x \mid \theta t) \text{ is Poisson, and}$$

$$\theta t = \left(\frac{(PFFH)(UF)(QPA)(1 - \underline{RIP})}{(NB)(\underline{MTBF})}\right) \left((\underline{RTS})(BRCT) + (\underline{NRTS})[(OSTCON)(1 - OS) + (OSTOS)(OS)]\right)$$

C_2 = cost of on-equipment maintenance

= (total mean number of failures) x (average on-equipment repair cost per failure)

$= [(TFFH)(UF)(QPA)/\underline{MTBF}] \times [\underline{PAMH} + (\underline{RIP})(\underline{IMH}) + (1 - \underline{RIP})(\underline{RMH})] \times BLR$

C_3 = cost of off-equipment maintenance

= (total mean number of off-equipment repairs) x (average cost per off-equipment repair)

$= [(TFFH)(UF)(QPA)(1 - \underline{RIP})/\underline{MTBF}] \times [(\underline{RTS})(\underline{BMH})(BLR + BMR) + (\underline{NRTS})(\underline{DMH})(DLR + DMR)]$

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*Where possible, definitions and symbology appearing in this typical LSC model framework are identical to those used in the Air Force Logistic Command's LSC Model. This model is described in reference 5. A further assessment of the model as a tool for comparison of competing systems, trade-off analysis, program cost and performance tracking, and trend evaluation can be found in reference 2.

DRCT	=	the average depot repair cycle time in months, i.e., the elapsed time for a NRTS item from removal of the failed item until it is made available to depot serviceable stock (G)
IMH	=	the average number of man-hours to perform corrective maintenance of the item in place or on line including fault isolation, repair, and verification (C)
MTBF	=	the mean time between failures in operating hours of the item in the operational environment (C)
NB	=	the number of operating locations (G)
NRTS	=	the fraction of removed items expected to be returned to depot for repair (= 1 - RTS) (C)
OS	=	the fraction of the total force deployed to overseas locations (G)
OSTCON	=	the average order and shipping time within the CONUS (G)
OSTOS	=	the average order and shipping time to overseas locations (G)
PAMH	=	the average number of man-hours expended in place on the complete system for preparation and access of the item; for example, jacking, unbuttoning, removal of other units, and hookup of support equipment (C)
PFFH	=	the peak force flying hours, i.e., the expected total fleet flying hours for one month during the peak usage period (G)
QPA	=	the quantity of like items within the parent system, i.e., "quantity per application" (C)
RIP	=	the fraction of item failures which can be repaired in place or on line (C)
RMH	=	the average number of man-hours to isolate a fault, remove and replace the item, and verify restoration of the system to operational status (C)
RTS	=	the fraction of removed items expected to be repaired at the base (= 1 - NRTS)
STK	=	the stock level of the item at each base (M)
TFFH	=	the expected total force flying hours over the program inventory usage period (the projected life of the item) (G)

II.1 (cont.)

UC = the negotiated unit cost of a spare item as of the end of the verification test (C)

UF = the ratio of operating hours to flying hours for the item (C)

XBO(i,θt) = expected number of backorders of the item at a given base as a function of the stock level, i, and the mean rate of demands, θt, for the item (M)

II.1 (cont.)

The nine underlined parameters (MTBF, RIP, RTS, NRTS, PAMH, IMH, RMH, BMH, and DMH) reflect hardware logistics characteristics over which the contractor has a degree of control through his design engineering process. These parameters are typically targeted by the Contractor in the equipment proposal. Let the following nine quantities be defined:

1. MTBF´ = the Contractor's target value for item MTBF as reflected in the Contractor's proposal and the LSCC

2. RIP´ = the Contractor's target value for item RIP rate as reflected ....

3. through 9. defined analogously with respect to RTS, NRTS, PAMH, IMH, RMH, BMH, and DMH in II.1 above.

II.2

Subsequently, they are estimated or "measured" during a field verification test, which typically covers a period of from 3,000 to 6,000 hours of operation of the equipment in its field environment. Let the following nine quantities be defined:

1. $\widetilde{MTBF}$ = the estimate of the item mean time between failures based on verification test failure data

2. $\widetilde{RIP}$ = the estimate of the fraction of failures of the item repaired in place based on verification test data

3. through 9. defined analogously with respect to RTS, NRTS, PAMH, IMH, RMH, BMH, and DMH in II.1.

II.3

Virtually all remaining model parameters (e.g., NB = number of bases to which the equipment will be deployed) describe the environment in which the equipment will operate and be maintained, and are supplied by the Government with the model framework.[2]

THE TARGET LOGISTIC SUPPORT COST

The negotiated contract includes a target LSC (TLSC) which reflects the LSC impact of the various Contractor-targeted equipment logistics parameters. We now define the target logistic support cost relative to the Figure 1 framework. Let

$$\begin{aligned} \text{TLSC} &\equiv \text{target logistic support cost} \\ &= C_1' + C_2' + C_3', \end{aligned} \qquad \text{II.4}$$

where C_i', i=1,2,3, is defined by the expression for C_i in Figure 1, except with those parameters defined by II.1 and underlined in the expression for C_i replaced by their corresponding <u>target</u> values defined by II.2. This target exists in addition to other contractual goals and targets such as a Design-to-Cost (DTC) goal on average unit flyaway cost. It is an assessment by the contractor of how well he feels his equipment will perform in the operational environment in terms of logistic support costs.

The incentive provisions of the LSCC are defined relative to the TLSC as opposed to the individual parameter targets, II.2. The parameter targets simply serve as a basis for development of the TLSC. A key objective underlying this approach is to increase the Contractor's design flexibility. For example, suppose the Contractor discovers after he is well into his design

[2]In most recent LSCC applications, one other critical equipment parameter, unit cost (UC) of spares, has also been targeted by the Contractor to be subsequently "measured" in the field. However, the "measured" value in these cases is simply the negotiated unit cost of spare items as of the end of the verification test. Hence, it is clearly not estimated in the statistical sense in which the other targeted parameters are estimated.

effort that he cannot achieve the MTBF target to which he committed himself in the contract (II.2.1) without initiating a large reliability improvement program at considerable cost to himself (and to the Government in the case of a cost-type contract). If the CMF suggests that this MTBF reduction can be offset by decreasing the projected percentage of failures requiring depot repair, i.e., reducing the NRTS rate, and if this alternative can be carried out through a less costly design change than a reliability improvement program, then the Contractor is encouraged to take it under the terms of the LSCC.[3]

The TLSC reflects interactions among logistics parameters that would not be captured by individual R&M targets. For example, equation C_1 in Figure 1 reflects the dramatic impact of a low MTBF-high NRTS combination on LSC much more realistically than separate targets on reliability and repair level would. The TLSC also provides a perceptual advantage by reflecting logistic support impacts of equipment characteristics in terms of dollars. This facilitates a more effective analysis of trade-offs between acquisition cost and LSC impacts of equipment design expenditures. Finally, the development of a competitive TLSC requires the Contractor to examine the relative effects of the various equipment logistics parameters over which he has control on LSC. As he gains experience in developing a competitive TLSC, he should become expert at trading off within the LSC model framework, as well as between LSC and costs of production, so as to converge to a minimum LCC design.

THE MEASURED LOGISTIC SUPPORT COST

The second key element of the LSCC concept is a verification test in the operational environment in which estimates of logistics parameters subject to

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[3] It is assumed in this illustration that the unforeseen decrement in reliability will not violate other terms of the contract, e.g., targets with respect to mission reliability, weapon system availability, etc.

verification are compiled from operational test data. These estimates are inserted into the LSCC cost element framework in place of the Contractor's targets, resulting in a new LSC figure known as the measured logistic support cost (MLSC). The MLSC is then compared to the TLSC and various decisions with respect to contract awards or remedies can be largely influenced by this comparison. We now define the measured logistic support cost relative to the Figure 1 framework. Let

$$\begin{aligned} \text{MLSC} &\equiv \text{measured logistic support cost} \\ &= \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3, \end{aligned} \tag{II.5}$$

where $\tilde{C}_i$, $i = 1,2,3$, is defined by the expression for C_i in Figure 1, except with those parameters defined by II.1 and underlined in the expression for C_i replaced by their corresponding <u>estimated</u> values defined by II.3.

If the MLSC is less than the TLSC, suggesting that the Contractor has more than achieved his LSC target, either of two types of positive adjustment can be used: (1) an award fee, the amount of which (up to a maximum specified in the LSCC) is determined by the Government as a function of the difference between the MLSC and the TLSC, and other logistic support cost performance criteria; or (2) an upward price adjustment, the simplest form of which is payment of a higher purchase price per unit. In the latter case, the amount of price adjustment is usually some function of the difference between the MLSC and TLSC and must be specified by a schedule appearing in the LSCC.[4]

If the MLSC exceeds the TLSC by some small amount, say 5-10%, the Contractor is usually given the benefit of the doubt and no contractual remedies with respect to LSC are required. However, a value of MLSC larger than some

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[4]One recent Air Force procurement incorporating this price adjustment provision is the ARC-164 UHF radio (managed by the Aeronautical Systems Division at Wright-Patterson AFB, Ohio), whereas the F-16 aircraft LSCC makes extensive use of the award fee provision.

threshold value, e.g., 1.25 x TLSC, is typically regarded as sufficient evidence that the LSC performance of the Contractor's equipment is inadequate. For MLSC values in this region, one or more of the following three forms of Contractor remedy is usually called for by the LSCC:

1. Exercise of a correction of deficiencies (COD) clause, the terms of which specify that the Contractor must improve the LSC performance of the equipment to the satisfaction of the Government. Costs of these improvements may be shared by the Contractor and the Government depending on the type of contract specifying this agreement.[5]

2. Reduction of the purchase price in the form of reduced fee or a reduced price per item. In either case, the price adjustment may be a function of the difference between the MLSC and TLSC, the form of which is specified in the LSCC.

3. A requirement for the Contractor to provide, at no additional cost to the Government, additional spare equipment items to offset the indicated LSC overrun. Here, a function relating the spares requirement to the difference between TLSC and MLSC is typically included in the LSCC.[6]

STATISTICAL RISK VERSUS TECHNICAL RISK

Both the Contractor and the Government are exposed to technical risk and statistical risk in the LSCC. In order to understand the nature of these risks, it is helpful to again consider the MLSC. Recall that the MLSC is a

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[5]This COD clause is a departure from COD clauses historically used with respect to operational performance. For example, the interpretation of the term "deficiency" is broader than and considerably different from the traditional interpretation. A deficiency with respect to LSC performance is defined to occur when the MLSC exceeds the TLSC to an unacceptable extent. This deficiency need not be defined relative to one aspect of LSC performance such as mean time between failures (MTBF) or mean time to repair (MTTR). Rather, a deficiency is defined to occur when poor performance relative to one or more of these logistics performance criteria combine to cause the unacceptably high MLSC. In addition, the period for notification to the Contractor concerning a deficiency is longer than traditional COD notification periods and the Contractor is permitted much greater flexibility in his corrective action.

[6]A further discussion of the use of positive and negative incentives in the LSCC can be found in Reference 12.

function of Government-supplied constants and estimates of Contractor targeted parameters such as MTBF, BMH, and NRTS. These estimates, e.g., $\widetilde{MTBF}$, $\widetilde{BMH}$, and $\widetilde{NRTS}$, have variability. Hence, the MLSC also has variability. Let p(MLSC) represent the probability distribution of the MLSC. In addition, let MMLSC represent the mean. The MLSC is an estimate of some real, underlying LSC value, say RLSC, defined in terms of the CMF. If the MLSC is unbiased, then its mean is equal to this value, i.e., MMLSC = RLSC. The particular definitions of the parameter estimators, e.g., $\widetilde{MTBF}$, $\widetilde{RTS}$, etc., determine whether or not the property of unbiasedness exists.

Statistical risk in the context of the LSCC is the risk of making incorrect award/remedy decisions because of differences between the MLSC and RLSC. In other words, it is the risk to both a Contractor and the Government of making incorrect award/remedy decisions because of variability of the MLSC. Technical risk, on the other hand, is risk to both the Contractor and the Government due to variability in the possible values that the underlying value, RLSC, might take on relative to the TLSC. This risk is more a function of the Contractor's technical ability to deliver an item of equipment whose LSC characteristics are close to a preassigned target. This technical ability is, in turn, a function of the state of the art of the proposed equipment, the efficiency of the Contractor's manufacturing processes, the realism of the proposed development schedule, etc.[7]

Technical risk and statistical risk should be addressed separately in structuring an LSCC. Both risks can have a significant impact on the effectiveness of the LSCC as an incentive transmitting tool.

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[7]This report will not address the topic of technical risk in detail. It should be noted, however, that assessing technical risk can be most difficult. Few useful quantitative measures of technical risk have been developed to date. Consequently, most technical risk assessments are made on a qualitative basis. Two useful references on this topic are References 24 and 25.

ANALYSIS PARAMETERS AND OBJECTIVES

A detailed treatment of the statistical risk implied by the LSCC can be undertaken in terms of the following four parameters:

1. Statistical risk to the Contractor.
2. Statistical risk to the Government.
3. Length of the verification test period in total weapon system operating hours, T.
4. An MLSC threshold above which a negative contract adjustment is required.[8]

The MLSC threshold is some given dollar amount larger than the TLSC. Let this quantity be called the remedy threshold, denoted by RT. It is convenient to be able to express the remedy threshold in terms of some factor (>1) times the TLSC. Let the expression TF, reflect this threshold factor. By this convention,

$$RT = TF \times TLSC \qquad \text{II.6}$$

The objectives of the anlaysis described in the remainder of this report are twofold:

1. to develop a quantitative measure of the variability of the MLSC distribution, p(MLSC), and to determine its properties, and
2. to discuss the importance of these properties regarding the overall process of structuring and managing the LSCC.

A HYPOTHESIS TESTING APPROACH

The essential elements of a typical LSCC can be characterized in terms of a hypothesis testing framework of classical statistics. First, the development of a TLSC by the Contractor in his equipment proposal can be interpreted

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[8] The treatment of MLSC thresholds beyond which positive adjustments such as the disbursement of award fees are required is very similar to the treatment of negative adjustments. The analysis in this report will be carried out only with respect to the negative adjustment. Extension of the mathematical results to the positive adjustment case is straightforward.

as an assertion by the Contractor that the mean MLSC reflected by his proposed equipment will be less than or equal to the TLSC, i.e.,

$$\text{MMLSC} \leq \text{TLSC}.^{9} \qquad \text{II.7}$$

This is the null hypothesis, H_0. The LSCC also addresses the possibility that H_0 may be false, i.e., that

$$\text{MMLSC} > \text{TLSC}. \qquad \text{II.8}$$

This is the alternative hypothesis, H_A. The basic decision rule incorporated in the LSCC is (1) to accept H_0 (seek no negative contract adjustments) if the MLSC does not exceed the remedy threshold, i.e., if

$$\text{MLSC} \leq \text{TF} * \text{TLSC} \qquad \text{II.9}$$

or (2) to reject H_0 (seek the remedy or negative adjustment) if the threshold <u>is</u> exceeded, i.e., if

$$\text{MLSC} > \text{TF} * \text{TLSC}. \qquad \text{II.10}$$

The interval from the TLSC to RT = TF x TLSC represents a margin of safety in the Contractor's favor.

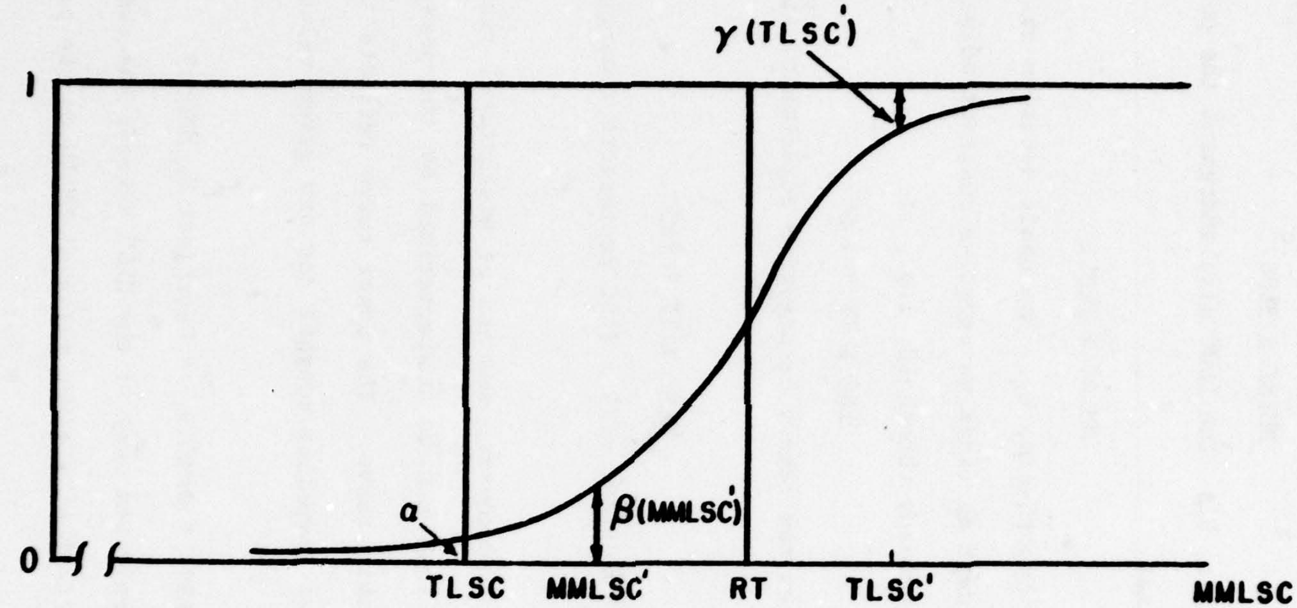
Given this set of hypotheses, amounts of statistical risk reflected by the field verification test can be characterized by the power curve of the test. Figure 2 shows this curve. The power curve reflects the probability that H_0 will be rejected (remedies sought) for any given value of the MMLSC, i.e.,

$$\text{Power(MMLSC)} \equiv \beta(\text{MMLSC}) = \Pr(\text{reject } H_0 | \text{MMLSC}) \qquad \text{II.11}$$

The hypothesis, H_0, is rejected only if the MLSC exceeds the remedy threshold. Hence, the power of the test at a given value of MMLSC is the probability that

[9] The development of this hypothesis testing framework assumes that the MLSC is an unbiased estimator of RLSC, i.e., that MMLSC = RLSC. Unbiasedness is discussed further in Chapter IV (and Lemma A6).

Figure 2. THE POWER CURVE OF THE TEST



WHERE

$\beta(\text{MMLSC}) = P_r(\text{REJECT } H_0 \mid \text{MMLSC})$

$\alpha = \beta(\text{TLSC}) = P_r(\text{REJECT } H_0 \mid H_0 \text{ TRUE}) = P_r(\text{TYPE I ERROR})$

$\gamma(\text{TLSC}') = 1 - \beta(\text{TLSC}') = P_r(\text{ACCEPT } H_0 \mid H_0 \text{ FALSE}) = P_r(\text{TYPE II ERROR})$

the MLSC will exceed the remedy threshold when the Contractor's equipment is actually characterized by that value of MMLSC, i.e.,

$$\beta(\text{MMLSC}) = \Pr(\text{MLSC} > \text{TF} * \text{TLSC} \mid \text{MMLSC}) \qquad \text{II.12}$$

For values of MMLSC to the left of the remedy threshold, the height of the power curve measures the probability that the verification test will yield a decision to seek Contractor remedies when, indeed, the value of MMLSC, if known, would have led to the opposite decision. The height of the power curve at MMLSC = TLSC, labelled α in classical statistics, represents the statistical risk to the Contractor under the LSCC. It reflects the probability of seeking remedies (rejecting the Contractor's equipment) when indeed, the equipment is adequate (H_0 true). Rejecting H_0 when MMLSC = TLSC is defined in classical statistics as the Type I error associated with the test. Hence, we can say

$$\begin{aligned} \alpha \equiv \beta(\text{TLSC}) &= \Pr(\text{reject } H_0 \mid H_0 \text{ true}) \\ &= \Pr(\text{MLSC} > \text{TF} \times \text{TLSC} \mid \text{MMLSC} = \text{TLSC}) \qquad \text{II.13} \\ &= \Pr(\text{Type I error}) \end{aligned}$$

For a value of MMLSC to the right of the remedy threshold, the quantity, $(1-\beta(\text{MMLSC}))$ measures the probability that the Contractor's equipment will be accepted when, indeed, it should be rejected (because the (unknown) MMLSC is greater than the remedy threshold). In order to develop an expression for Government risk analogous to α, we define TLSC′ to be the Government's "rejection target," i.e., a specified value of MMLSC (>RT) at which the Government wants to ensure a high probability of rejection. The statistical risk to the Government under the LSCC is defined to be the probability, henceforth called

$\gamma(TLSC')$, of getting an MLSC value which suggests acceptance of the Contractor's equipment when indeed, he has actually overrun his target to the extent that MMLSC = TLSC', i.e.,

$$\left.\begin{aligned} \gamma(TLSC') &= 1 - \beta(TLSC') = 1 - Pr(MLSC > TF \times TLSC \mid MMLSC = TLSC') \\ &= Pr(MLSC < TF \times TLSC \mid MMLSC = TLSC'). \end{aligned}\right] \quad \text{II.14}$$

This type of decision error is called the Type II error associated with the test. Therefore,

$$\gamma(TLSC') = Pr(\text{Type II error}). \quad \text{II.15}$$

ORGANIZATION OF REPORT

In the next chapter, we develop the MLSC estimator and describe development assumptions. Chapters IV and V present the mean and variance properties of the estimator and their impact on its power function, β(MMLSC). Chapter VI describes a Monte Carlo simulation model which can be used to approximate the estimator's variance characteristics in the multiple equipment item case. Finally, in Chapter VII, we examine the potential for further research into the LSCC.

III. DEVELOPMENT OF THE MLSC ESTIMATOR

VARIABLES TO BE OBSERVED

Given that the CMF in Figure 1 is used as the basis for development of the TLSC for a given equipment item, the MLSC can be calculated as a function of the observed quantities,

$X_1(T)$ = the number of item failures during the verification test interval, T, requiring in-place repair,

$X_2(T)$ = the number of item failures in time T requiring base repair,

$X_3(T)$ = the number of item failures in time T requiring depot repair (assuming one depot),

Y_{1k} = the number of man-hours for the k'th preparation and access, $k = 1,2,\ldots, X(T)$ (where we note that $X(T) = X_1(T) + X_2(T) + X_3(T)$),

Y_{2k} = the number of man-hours for the k'th in place repair, $k = 1,2,\ldots, X_1(T)$,

Y_{3k} = the number of man-hours for the k'th removal and replacement, $k = 1,2,\ldots, X_2(T) + X_3(T)$,

Y_{4k} = the number of man-hours for the k'th base repair, $k = 1,2,\ldots, X_2(T)$,

Y_{5k} = the number of man-hours for the k'th depot repair, $k = 1,2,\ldots, X_3(T)$,

III.1a

where it is implicitly assumed by the LSCC that

$$\begin{aligned}
E(X_1(T)) &= RIP \cdot T/MTBF, \\
E(X_2(T)) &= RTS \cdot (1-RIP) \cdot T/MTBF, \\
E(X_3(T)) &= NRTS \cdot (1-RIP) \cdot T/MTBF, \\
E(X(T)) &= (RIP + RTS \cdot (1-RIP) + NRTS \cdot (1-RIP)) \\
&\quad \cdot T/MTBF \\
&= T/MTBF = \lambda T \text{ (where } \lambda \equiv 1/MTBF),
\end{aligned} \qquad \text{III.1b}$$

$$
\left.\begin{aligned}
E(Y_{1k}) &= PAMH,\ k = 1,2,\ldots,\ X(T), \\
E(Y_{2k}) &= IMH,\ k = 1,2,\ldots,\ X_1(T), \\
E(Y_{3k}) &= RMH,\ k = 1,2,\ldots,\ X_2(T) + X_3(T), \\
E(Y_{4k}) &= BMH,\ k = 1,2,\ldots,\ X_2(T), \\
E(Y_{5k}) &= DMH,\ k = 1,2,\ldots,\ X_3(T),
\end{aligned}\right| \quad \text{III.1b cont'd}
$$

(see II.1) and all the observed quantities in III.1a have finite variances.

All but the last of these eight variables can be observed at the base level. In several Air Force applications of the LSCC currently in progress, these observations are made from Air Force 66-1 records subject to a base level audit. The parameter DMH (depot repair hours per failure) has historically not been made subject to verification for two reasons. First, it is difficult and costly to set up and manage special verification procedures at the depot. Second, since the items involved frequently have low NRTS rates (e.g., .05-.10), few failures over a verification test period of just a few thousand hours are sent to depot for repair. Thus, there are few observations upon which to build an estimate of DMH (i.e., $X_3(T)$ is small). In such a case, the parameter, DMH, is simply defined as a constant in the contract and the set of observations, Y_{5k}, $k = 1,2,\ldots,X_3(T)$, is left undefined.[1]

---

[1] It would appear that not subjecting the target value for DMH to verification would tend to encourage the Contractor to concentrate his design effort more toward influencing base level equipment repair parameters (e.g., reducing BMH) in order to meet the TLSC at the expense of a possible increase in the ultimate value of DMH. In some recent LSCC applications, an attempt has been made to offset this tendency by including a constraint on average (over bases and depot) manpower cost in the LSCC. Such a constraint might have the form,

$$RTS * BMH + NRTS * DMH \leqq \text{constant}. \qquad \text{III.2}$$

This constraint is clearly aimed in the right direction. However, while the estimated version of the constraint may be fulfilled, it is difficult to ascertain the confidence with which the original form is fulfilled because the variability of the quantity,

$$\widetilde{RTS} * \widetilde{BMH} + \widetilde{NRTS} * DMH, \qquad \text{III.3}$$

DEVELOPMENT OF THE ESTIMATOR

It is instructive to characterize the measured logistic support cost (MLSC) estimate as defined in general in II.5 in terms of the particular observed variables defined above. In most recent applications of the LSCC, the various targeted parameters in the CMF have been estimated essentially by sample means, i.e.,

$$
\left.
\begin{aligned}
\widetilde{MTBF}(T) &= \text{estimated MTBF} \\
&= T/X(T) \quad \text{where, from above, } T = \text{test length in flying hours and } X(T) = \text{number of failures occurring during the T hours,}^{2} \\
\widetilde{RIP}(T) &= \text{estimated probability that a failure is repaired in place} \\
&= X_1(T)/X(T), \\
\widetilde{RTS}(T) &= \text{estimated probability that a removed item is repaired at base} \\
&= X_2(T)/(X_2(T) + X_3(T)), \\
\widetilde{NRTS}(T) &= \text{estimated probability that a removed item is repaired at depot} \\
&= X_3(T)/(X_2(T) + X_3(T)) \quad (= 1 - \widetilde{RTS}),
\end{aligned}
\right] \qquad \text{III.4}
$$

(the estimate of the left-hand-side of III.2) is not readily known and its mean isn't necessarily equal to RTS * BMH + NRTS * DMH. Hence, in any given case, the constraint may not be achieving its desired effect. These questions are worthy of further research.

[2] We assume $X(T) \geq 1$ for the remainder of this report. Reference 26 describes some approaches to modifying this estimator to handle the zero failure case ($X(T)=0$) if, indeed, T is small enough so that this case is likely to occur. The effects of using such a modified estimator on the distribution of the MLSC estimate appearing in Expression III.5 below have not been examined.

$\widetilde{PAMH}(T)$ = estimated mean number of man-hours for preparation and access

$$= (1/X(T)) \sum_{k=1}^{X(T)} Y_{1k},$$

$\widetilde{IMH}(T)$ = estimated mean number of man-hours for in place repair

$$= (1/X_1(T)) \sum_{k=1}^{X_1(T)} Y_{2k},$$

$\widetilde{RMH}(T)$ = estimated mean number of man-hours for removal and replacement

III.4 cont'd.

$$= (1/(X_2(T)+X_3(T))) \sum_{k=1}^{X_2(T)+X_3(T)} Y_{3k},$$

$\widetilde{BMH}(T)$ = estimated mean number of man-hours for base level repair

$$= (1/X_2(T)) \sum_{k=1}^{X_2(T)} Y_{4k},$$

$\widetilde{DMH}(T)$ = estimated mean number of man-hours for depot level repair

$$= (1/X_3(T) \sum_{k=1}^{X_3(T)} Y_{5k}.$$

Though variants of these definitions might be more appropriate in certain cases, these particular definitions are intuitively reasonable and have the quality of simplicity.[3] Substituting them into the general definition of

[3] A variety of alternative estimator definitions could be entertained. For example, MTBF could be defined as

$$\widetilde{MTBF}(T) = \sum_{i=1}^{X(T)} t_i/X(T), \quad \text{(where } t_i = \text{the time interval between the (i-1)'th and i'th failure, } i = 1,2,\ldots, X(T))$$

which of course has slightly different statistical properties than $T/X(T)$. (If we assume that $X(T)$ is Poisson with parameter λT where $\lambda = 1/MTBF$, then $T/X(T)$, or equivalently, $X(T)/T$ as an estimator of λ is highly justified since $X(T)/T$ is both the maximum likelihood and unbiased estimate of λ (see references 11 and 26)).

MLSC in II.5 and cancelling terms where appropriate yields the following expression:

$$\begin{aligned}
MLSC(T) = {} & \underbrace{K_0 \, \widetilde{STK}(\widetilde{\theta t}, EBO)}_{\text{Base Repair Pipeline Spares}} + \underbrace{(K_1/T)\, X_3(T)}_{\text{Depot Repair Pipeline Spares}} \\
& + \underbrace{(K_2/T) \sum_{k=1}^{X(T)} Y_{1k}}_{\text{Prep. and Access Man-hours}} + \underbrace{(K_2/T) \sum_{k=1}^{X_1(T)} Y_{2k}}_{\text{In Place Repair Man-hours}} \\
& + \underbrace{(K_2/T) \sum_{k=1}^{X_2(T)+X_3(T)} Y_{3k}}_{\text{Remove and Replace Man-hours}} + \underbrace{(K_3K_4/T) \sum_{k=1}^{X_2(T)} Y_{4k}}_{\text{Base Level Repair Man-hours}} + \underbrace{(K_3K_5/T) \sum_{k=1}^{X_3(T)} Y_{5k}}_{\text{Depot Level Repair Man-hours}}
\end{aligned} \qquad \text{III.5a}$$

where $\widetilde{STK}(\widetilde{\theta t}, EBO)$ is the minimum value of i such that

$$\begin{aligned}
XBO(i, \widetilde{\theta t}) &= \sum_{x>i} (x-i)\, p(x|\widetilde{\theta t}) \leq EBO, \\
p(x|\widetilde{\theta t}) &= e^{\widetilde{\theta t}} \frac{(\widetilde{\theta t})^x}{x!} \text{ (Poisson), and} \\
\widetilde{\theta t} &= K_6 X_2(T) + K_7 X_3(T),
\end{aligned} \qquad \text{III.5b}$$

where the phrase below each term in III.3a describes the element of cost estimated, and where we have grouped the various constants that appear in the general expression for MLSC (II.5) for simplicity of exposition as shown below:[4]

$$\left.\begin{aligned}
K_0 &= NB * UC \\
K_1 &= PFFH * UF * QPA * DRCT * UC \\
K_2 &= TFFH * UF * QPA * BLR
\end{aligned}\right] \qquad \text{III.5c}$$

[4]Note that T ≡ verification test length in hours while t ≡ the weighted (between base and depot) pipeline time for the item and x ≡ the number of demands for the item on the base stock during the time interval t. Hence, the estimate, $\widetilde{\theta t}$, of the quantity, θt (see definition in Figure 1), is a function of T like all the other terms of III.5a.

$$
\left.\begin{aligned}
K_3 &= TFFH * UF * QPA \\
K_4 &= BLR + BMR \\
K_5 &= DLR + DMR \\
K_6 &= PFFH * QPA * UF * BRCT/(NB * T) \\
K_7 &= PFFH * QPA * UF * (OSTCON*(1-OS)+OSTOS*OS)/(NB*T)
\end{aligned}\right] \quad \text{III.5c (cont'd)}
$$

Expression III.5 forms the basis for the statistical analysis documented in this report. It is a most important expression because it expresses the MLSC, whose value can potentially influence a very large dollar amount of both Contractor and Government expenditures, directly as a function of reliability and maintainability data values observed in the field. Relative to the general expression for the MLSC in II.5, it is fairly simple in form. This occurs because the general expression contains several products of observed variables, in which numerous factors cancel with one another when specific estimator definitions are substituted. For example, the base level repair man-hour cost estimator term in $\tilde{C}_3$ simplifies as follows:

$$
\left.\begin{aligned}
& K_3K_4(1-\widetilde{RIP})(\widetilde{RTS})(\widetilde{BMH})/\widetilde{MTBF} \\
&= K_3K_4((X_2(T)+X_3(T))/X(T))*(X_2(T)/(X_2(T)+X_3(T))) \\
&\qquad *\left(\sum_{k=1}^{X_2(T)} Y_{4k}/X_2(T)\right)*(X(T)/T) \\
&= (K_3K_4/T)\sum_{k=1}^{X_2(T)} Y_{4k}.
\end{aligned}\right] \quad \text{III.6}
$$

PROPERTIES OF THE ESTIMATOR DISTRIBUTION

Upon the adoption of a few important assumptions regarding the distributions of some of the observed variables, some very useful properties can be attributed to the MLSC estimator distribution. More specifically, exact

expressions for the mean and variance of the MLSC, and the power curve, β(MMLSC), can be formulated and, in many cases, numerically computed or approximated. The major requisite assumptions are described below.

Poisson Failures

The number of equipment failures in time T, X(T), is assumed to have a Poisson distribution with parameter λT, where $\lambda = 1/MTBF$ by III.1b. This assumption is virtually always made in the modeling of failure behavior of military avionics hardware. It forms the basis for Military Standard 781B which is used throughout the military services for R&D testing of avionics equipment. The assumption suggests that failures are truly random or in other words, that the equipment has matured in its failure behavior and is not wearing out. In the case of the LSCC, it is readily defensible on two grounds. First, the verification test under the LSCC is not usually begun until the equipment has been in the field for some time, e.g., six months to a year, during which initial transient failure modes can be isolated and removed. Second, even if the equipment has not matured, and hence, the number of failures in time T of each individual equipment item in the fleet is not Poisson, the number of failures in time T for the whole fleet of items is approximately Poisson. This occurs because the sequence of time intervals between failures for the entire fleet is a superposition of renewal processes. Hence, the Poisson assumption is valid here since the definition of X(T) does not require that the identity of each individual item be maintained and that each individual item failure rate be tracked.

Poisson Arrivals for Repair

The Poisson failure assumption leads to a further useful result. Let

p_1 ≡ RIP = probability of repair in place given a failure,

p_2 ≡ RTS (1-RIP) = (probability of repair at base given a failure and removal) * (probability of removal given a failure), III.7

= probability of removal and repair at base given a failure[5],

$p_3 \equiv$ NRTS (1-RIP) = probability of removal and repair at depot given a failure, III.7 cont.d

and recall that

$\lambda = 1/MTBF$ = failure rate of the item.

These definitions assume that the p_i's are independent of the state of the system, i.e., that each of the probabilities is independent of the preceding and succeeding sequence of failures and subsequent repair actions.

If the number of failures in time T, $X(T) = X_1(T) + X_2(T) + X_3(T)$ is Poisson with parameter λ, it can be shown (see Lemma A1) that (1) the number of failures in time T requiring the ith type of repair, $X_i(T)$, i = 1,2,3, is also Poisson with parameter $p_i\lambda T$ and (2) $X_1(T)$, $X_2(T)$, and $X_3(T)$ are mutually independent.

Independence: The Compound Poisson Process

Note that each of the last five terms in III.5a is equal to a constant times a variable number of man-hour observations. It is quite reasonable to assume with respect to the second of these five terms,

$$(K_2/T) \sum_{k=1}^{X_1(T)} Y_{2k}, \qquad \text{III.8}$$

that the family of in-place repair man-hour observations, Y_{2k}, k = 1,2,..., $X_1(T)$ are independent, identically distributed random variables. It is also reasonable to assume that the number of arrivals for repair, $X_1(T)$, forming the limit of this sum and the sequence of man-hour observations, Y_{2k}, being summed are independent. In other words, the rate of demands for repair is not

[5] If A = a failure event, B = a removal event, and C = a base repair event, then 1-RIP = $P(B|A)$ and RTS = $P(C|AB)$. Hence, $p_2 = P(C|AB)\,P(B|A) = (P(ABC)/P(AB))*(P(AB)/P(A)) = P(ABC)/P(A) = P(BC|A)$.

influenced by the man-hours per repair and vice versa. Analogous assumptions can clearly be made with respect to the fourth and fifth of these five terms.

Given these two assumptions, and the Poisson arrival assumption, each of these three sums is a Compound Poisson process having well known properties.[6] Furthermore, the first sum,

$$\sum_{k=1}^{X(T)} Y_{1k}, \qquad \text{III.9}$$

can be separated into three compound Poisson sums,

$$\sum_{k=1}^{X_1(T)} Y_{1k}^{(1)} + \sum_{k=1}^{X_2(T)} Y_{1k}^{(2)} + \sum_{k=1}^{X_3(T)} Y_{1k}^{(3)}, \qquad \text{III.10}$$

with the Y_{1k}'s appropriately reordered, as indicated by the superscript, such that the first sum includes only those $X_1(T)$ preparation and access (P&A) man-hour observations, $Y_{1k}^{(1)}$, that led to repair in place, the second sum includes those $X_2(T)$ P&A observations, $Y_{1k}^{(2)}$, that led to base repair, etc. The sum in the third of the last five terms in III.5a can be decomposed similarly into two compound Poisson sums.

Hence, MLSC(T) can be written as a linear combination of ten terms, i.e.,

$$\left.\begin{aligned}
MLSC(T) = {} & K_0\,\widetilde{STK}(\widetilde{\theta t}, EBO) + (K_1/T)\,X_3(T) + (K_2/T) \sum_{k=1}^{X_1(T)} Y_{1k}^{(1)} \\
& + (K_2/T) \sum_{k=1}^{X_2(T)} Y_{1k}^{(2)} + (K_2/T) \sum_{k=1}^{X_3(T)} Y_{1k}^{(3)} + (K_2/T) \sum_{k=1}^{X_1(T)} Y_{2k} \\
& + (K_2/T) \sum_{k=1}^{X_2(T)} Y_{3k}^{(2)} + (K_2/T) \sum_{k=1}^{X_3(T)} Y_{3k}^{(3)} + (K_3K_4/T) \sum_{k=1}^{X_2(T)} Y_{4k} \\
& + (K_3K_5/T) \sum_{k=1}^{X_3(T)} Y_{5k},
\end{aligned}\right] \qquad \text{III.11}$$

[6]The compound Poisson process is described in reference 19, pp. 128-131.

where the first has an as yet unknown distribution, the second is a constant times a Poisson random variable, and the last eight are each a constant times a compound Poisson sum.

Normality

Note that all of the compound Poisson sums in III.11 are not mutually independent because several sums share the same upper limit. Rewriting the expression with all such sums grouped accordingly yields

$$
\begin{aligned}
MLSC(T) = \; & K_0 \, \widetilde{STK}(\widetilde{\theta t}, EBO) + (K_1/T)\, X_3(T) + \sum_{k=1}^{X_1(T)} ((K_2/T) Y_{1k}^{(1)} \\
& + (K_2/T) Y_{2k}) + \sum_{k=1}^{X_2(T)} ((K_2/T) Y_{1k}^{(2)} + (K_2/T) Y_{3k}^{(2)} + (K_3 K_4/T) Y_{4k}) \\
& + \sum_{k=1}^{X_3(T)} ((K_2/T) Y_{1k}^{(3)} + (K_2/T) Y_{3k}^{(3)} + (K_3 K_5/T) Y_{5k})
\end{aligned}
\qquad \text{III.12}
$$

It is quite reasonable to assume that observations of man-hours to prepare and access (Y_{1k}), remove and replace (Y_{3k}), repair in place (Y_{2k}), repair at base (Y_{4k}), and repair at depot (Y_{5k}) are mutually independent. If so, then all terms in each of the three sums in III.12 are mutually independent. Furthermore, since the sums are clearly mutually independent, then all terms in the three sums are mutually independent.

This amount of independence is viewed as sufficient to invoke a generalized version of the Central Limit Theorem[7] in order to make the assumption that MLSC(T) is approximately normal. Neither the effects of skewness and discreteness in the distributions of the first two terms in III.12 nor the

[7]See reference 1 and reference 13, pp. 278-288.

effects of co-variances between these distributions and various of the compound Poisson processes is likely to be sufficient to negate the normality assumption.[8]

Let us now examine the power curve in view of the normality assumption. Recall from II.12 that it has the form,

$$\beta(MMLSC) = Pr(MLSC > TF * TLSC \mid MMLSC) \qquad III.13$$

or, with a substitution and small change of notation,

$$\beta(MMLSC) = Pr_{MMLSC}(MLSC > RT). \qquad III.14$$

Let σ^2_{MMLSC} denote the variance of MLSC when MMLSC is its underlying unknown mean. Subtracting MMLSC from both sides of the inequality in III.14 and dividing both sides by the standard deviation, $\sqrt{\sigma^2_{MMLSC}}$, gives

$$\beta(MMLSC) = Pr_{MMLSC}\left(\frac{MLSC-MMLSC}{\sqrt{\sigma^2_{MMLSC}}} > \frac{RT-MMLSC}{\sqrt{\sigma^2_{MMLSC}}}\right) \qquad III.15$$

Rewriting III.15 to reflect the assumption that MLSC is approximately normal, we have

$$\beta(MMLSC) = Pr_{MMLSC}\left(Z > \frac{RT-MMLSC}{\sqrt{\sigma^2_{MMLSC}}}\right) \quad \text{where } Z \sim N(0,1) \qquad III.16$$

Having made the normality assumption, the task of developing a power curve expression that can be numerically computed or approximated reduces to developing expressions for MMLSC and σ^2_{MMLSC} in terms of the underlying unknown equipment parameters MTBF, RIP, RTS, PAMH, etc.

---

[8]While this assumption has been validated using a Kolmogorov-Smirnov statistic in the case where the expression is summed repeatedly for several items (see reference 7, pp. 85-89]), a statistical analysis in the case of just a single equipment item as reflected by III.12 has not yet been undertaken and would be a useful exercise.

Because many of the properties of the first (base spares cost estimator) term in III.12 are highly complex, we first derive the mean and variance properties of MLSC assuming the first term is absent. We call this simplified estimator MLSC′, i.e.,

$$MLSC' \equiv MLSC - K_0 \, \widetilde{STK} \, (\tilde{\theta}t, EBO). \qquad III.17$$

The properties of this estimator not only provide a baseline upon which to evaluate the more complex MLSC estimator properties but also have special significance in themselves. The considerable statistical complexity of the base spares cost estimator term is due to the fact that the LSC model base spares cost term is formulated to provide a safety stock of base spares so that the expected number of base backorders can be held at a specified low level. If this term were simplified to capture only the mean number of required base spares as was the case in some earlier USAF editions of the LSC model, it would have essentially the same form as the second depot level spares term. Furthermore, with this simplification, the MLSC′ estimator could be extended to include a base spares cost estimator with no significant change in its statistical properties.

In Chapter IV, we develop the mean and variance of MLSC′ and in Chapter V, the mean and variance of MLSC.

IV. MEAN AND VARIANCE PROPERTIES OF MLSC´

MOMENTS OF THE SIMPLIFIED ESTIMATOR TERMS

Development of the means and variances of and covariances among the terms of the simplified estimator,

$$\left.\begin{aligned} MLSC'(T) = {} & (K_1/T)X_3(T) + (K_2/T)\sum_{k=1}^{X_1(T)} Y_{1k}^{(1)} + (K_2/T)\sum_{k=1}^{X_1(T)} Y_{2k} \\ & + (K_2/T)\sum_{k=1}^{X_2(T)} Y_{1k}^{(2)} + (K_2/T)\sum_{k=1}^{X_2(T)} Y_{3k}^{(2)} + (K_3K_4/T)\sum_{k=1}^{X_2(T)} Y_{4k} \\ & + (K_2/T)\sum_{k=1}^{X_3(T)} Y_{1k}^{(3)} + (K_2/T)\sum_{k=1}^{X_3(T)} Y_{3k}^{(3)} + (K_3K_5/T)\sum_{k=1}^{X_3(T)} Y_{5k} \end{aligned}\right] \qquad \text{IV.1}$$

is straightforward. For simplicity of exposition, let

$$\mu_1 \equiv PAMH,\ \mu_2 \equiv IMH,\ \mu_3 \equiv RMH,\ \mu_4 = BMH,\ \mu_5 = DMH, \qquad \text{IV.2}$$

so that from III.1b and IV.2,

$$E(Y_{ik}) = \mu_i,\ i=1,2,3,4,5. \qquad \text{IV.3}$$

Recall from Chapter III that

$$p_1 = RIP,\ p_2 = RTS(1-RIP),\ p_3 = NRTS(1-RIP) \text{ and } \lambda=1/MTBF, \qquad \text{IV.4}$$

so that by the Poisson assumption,

$$E(X_i(T)) = Var(X_i(T)) = p_i\lambda T,\ i = 1,2,3. \qquad \text{IV.5}$$

Also, let the finite variances of the five man-hour distributions be defined as

$$\left.\begin{aligned} Var\ Y_{1k} &= \sigma_1^2 \\ Var\ Y_{2k} &= \sigma_2^2 \\ Var\ Y_{3k} &= \sigma_3^2 \\ Var\ Y_{4k} &= \sigma_4^2 \end{aligned}\right] \qquad \text{IV.6}$$

$$\text{Var } Y_{5k} = \sigma_5^{\ 2}$$

for all appropriate values of k in each case.

Using these definitions and assumptions, it can be shown (see reference 19, pp. 128-131 and Lemmas A2-A5) that

$$E\left\{\sum_{k=1}^{X_i(T)} Y_{jk}\right\} = \mu_j p_i \lambda T, \qquad \text{IV.7a}$$

$$\text{Var}\left\{\sum_{k=1}^{X_i(T)} Y_{jk}\right\} = p_i \lambda T(\mu_j^2 + \sigma_j^2), \qquad \text{IV.7b}$$

$$\text{Cov}\left\{X_i(T), \sum_{k=1}^{X_i(T)} Y_{jk}\right\} = \mu_j p_i \lambda T, \qquad \text{IV.7c}$$

and

$$\text{Cov}\left\{\sum_{k=1}^{X_i(T)} Y_{mk}, \sum_{k=1}^{X_i(T)} Y_{nk}\right\} = \mu_m \mu_n p_i \lambda T, \quad m \neq n \qquad \text{IV.7d}$$

for appropriate combinations of i, j, m, and n.

Furthermore, it is reasonable to assume that each of the five distributions of man-hours to repair has a constant variance to mean ratio. Substantial military equipment maintainability testing experience supports this assumption and, in fact, indicates that the ratio falls within a fairly narrow range of values.[1] Accordingly, let

$$\sigma_j^{\ 2} = c_j \mu_j, \qquad j = 1, 2, \ldots, 5, \qquad \text{IV.8}$$

---

[1]Reference 7, pp. 70-73, makes the additional assumption, not required in our analysis, that maintenance man-hour distributions are lognormal, providing considerable evidence in support of this assumption from Category II maintainability test data. It makes the constant variance to mean ratio assumption in the context of the lognormal assumption and suggests a ratio value in the range of 1.0 to 1.5 based on the Category II test data.

where c_j, $j=1,2,\ldots 5$, is a constant estimated during the LSCC structuring process.[2] Making this substitution in IV.7b leads to

$$\mathrm{Var}\left|\sum_{k=1}^{X_i(T)} Y_{jk}\right| = p_i\lambda T(\mu_j^2 + c_j\mu_j) \qquad \text{IV.7b}'$$

for appropriate combinations of i and j. This reduction from a functional dependence on four unknown parameters (p_i, λ, μ_j, and σ_j^2) to three unknown parameters significantly helps to simplify analysis of the LSCC statistical properties.

<u>THE MEAN AND VARIANCE FUNCTIONS: IMPACT ON USEFULNESS OF THE LSCC</u>

Taking the mean and variance of IV.1 and substituting IV.7a, b′, c, and d as appropriate leads to the following expressions for the mean and variance of MLSC′:

$$\mathrm{MMLSC}' \equiv E(\mathrm{MLSC}') = \lambda(K_1p_3 + K_2\mu_1 + K_2p_1\mu_2 + K_2p_2\mu_3 + K_2p_3\mu_3 + K_3K_4p_2\mu_4 + K_3K_5p_3\mu_5) \qquad \text{IV.9}$$

$$\begin{aligned}\mathrm{Var}(\mathrm{MLSC}') = (\lambda/T)(&K_1^2p_3 + K_2^2c_1\mu_1 + K_2^2\mu_1^2 + K_2^2c_2p_1\mu_2 + K_2^2p_1\mu_2^2 + K_2^2c_3p_2\mu_3 \\ &+ K_2^2p_2\mu_3^2 + K_2^2c_3p_3\mu_3 + K_2^2p_3\mu_3^2 + K_3^2K_4^2c_4p_2\mu_4 + K_3^2K_4^2p_2\mu_4^2 \\ &+ K_3^2K_5^2c_5p_3\mu_5 + K_3^2K_5^2p_3\mu_5^2 + 2(K_1K_2p_3\mu_1 + K_1K_2p_3\mu_3 + K_1K_3K_5p_3\mu_5 \\ &+ K_2^2p_1\mu_1\mu_2 + K_2^2p_2\mu_1\mu_3 + K_2^2p_3\mu_1\mu_3 + K_2K_3K_4p_2\mu_1\mu_4 + K_2K_3K_5p_3\mu_1\mu_5 \\ &+ K_2K_3K_4p_2\mu_3\mu_4 + K_2K_3K_5p_3\mu_3\mu_5))\end{aligned} \qquad \text{IV.10a}$$

[2] Sensitivity analyses indicate that small differences among alternative values that c_j may be judged to have within the region of 1.0 to 1.5 have a very minor effect on the resulting statistical risk underlying the LSCC.

Also, let

$$\sigma^2_{MMLSC'} \equiv \underset{MMLSC'}{\mathrm{Var}}\ (MLSC') = \text{the variance of } MLSC' \ (\mathrm{Var}(MLSC')) \text{ given its underlying unknown mean is } MMLSC' \qquad \text{IV.10b}$$

It can be readily shown (see Lemma A6) that

$$E(MLSC') = RLSC' \ (\equiv C_1 + C_2 + C_3 - (M)(STK)(UC)) \qquad \text{IV.11}$$

i.e., that the particular choice, III.4, of estimators of the nine hardware logistics parameters, has resulted in MLSC′ being an unbiased estimator of the unknown underlying cost figure, RLSC′.

Also, note that $\sigma^2_{MMLSC'}$ is a higher order function of the nine logistics parameters, p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, μ_5, and λ than is MMLSC′ with the obvious result that for any given value of MMLSC′, there exists a <u>range</u> of possible values of $\sigma^2_{MMLSC'}$ and by III.16, a range of possible power function values, $\beta(MMLSC')$, at this point. The properties of this range have a pivotal impact on the usefulness of the LSCC.

Clearly, for any given value of MMLSC′, values of the nine hardware logistics parameters, not all necessarily realistic, can be found such that $\sigma^2_{MMLSC'} \to \infty$ and hence, $\beta(MMLSC') \to .5$. But the more important result is that if the variance can become very large or very small for combinations of unknown parameter values which reflect a realistic possible value of MMLSC′ <u>and are realistic values themselves</u>, then statistical risks to both the Contractor and Government become impossible to assess meaningfully, thereby compromising the legal enforceability of the LSCC and greatly diminishing its effectiveness as an incentive transmitting tool.

These conditions naturally lead one to examine the feasibility of individual bounds on certain of the logistics parameters in the context of the LSCC. Such bounds, if appropriately formulated as part of the LSCC structuring process, can be interpreted meaningfully either as additional contractor commitments or additional government constraints on logistics

performance. The mathematical result of such bounding is a bounded range of power function values for any given value of the unknown underlying MMLSC′. The size of this range is, in general, not easily calculated and, depending on how individual parameter bounds are formulated, may or may not be small enough to permit useful LSCC applications.

We now formulate certain bounds to confine the individual logistics parameters to ranges of realistic possible values and explore the properties of the range of variances that results.

A FRAMEWORK FOR EXPLORING THE VARIANCE RANGE

The problem of finding a maximum (or minimum) value of the variance, $\sigma^2_{MMLSC'}$, subject to MMLSC′ taking on a certain value and other appropriate constraints on the nine unknown hardware logistics parameters, can be formulated in terms of the nonlinear programming (NLP) framework below:

$$
\left.
\begin{array}{l}
\text{maximize (minimize) } \sigma^2_{MMLSC'} = f(p_1, p_2, p_3, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \lambda) \\
\text{subject to} \\
\quad \text{a.} \quad MMLSC' \equiv g(p_1, p_2, p_3, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \lambda) \\
\qquad\qquad = K \\
\quad \text{b.} \quad p_1 + p_2 + p_3 = 1 \\
\quad \text{c.} \quad 1. \quad LB_1 \le p_1 \le UB_1 \\
\qquad\quad 2. \quad LB_2 \le p_2 \le UB_2 \\
\qquad\quad 3. \quad LB_3 \le p_3 \le UB_3 \\
\qquad\quad 4. \quad LB_4 \le \mu_1 \le UB_4 \\
\qquad\quad 5. \quad LB_5 \le \mu_2 \le UB_5 \\
\qquad\quad 6. \quad LB_6 \le \mu_3 \le UB_6 \\
\qquad\quad 7. \quad LB_7 \le \mu_4 \le UB_7 \\
\qquad\quad 8. \quad LB_8 \le \mu_5 \le UB_8 \\
\quad (LB_i \ge 0,\ i = 1,2,\ldots,8)
\end{array}
\right]
\quad \text{IV.12}
$$

where f(·) and g(·) in IV.12 are defined to be IV.10 and IV.9, respectively.

Constraint IV.12a sets MMLSC′, the underlying unknown mean, equal to K, where K can take on any value on the domain of the power curve, β(MMLSC′). Constraint IV.12b requires the probabilities of (1) in-place repair, (2) removal and base repair, and (3) removal and depot repair to sum to one.

Constraints IV.12c reflect upper and lower bounds on the three repair mode probabilities, RIP, (1-RIP)RTS, and (1-RIP)NRTS, and the five man-hour distribution means, PAMH, IMH, RMH, BMH, and DMH.[3] These bounds must be determined subjectively and with care since small changes in their values do have a noticeable (but generally not prohibitive) impact on the size of the variance range.[4]

There is at least one potentially feasible interpretation of the role of Constraints IV.12c in the context of the LSCC. If K$\leq$TLSC, they can be interpreted as the Contractor's best engineering assessment of the regions in which the unknown equipment logistics parameters actually lie and in which the parameters can be varied in order to produce a value of RLSC′ $\leq$ TLSC. Accordingly these bounds could be provided by the Contractor along with his TLSC

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[3]Note that these upper and lower bounds on all hardware logistics parameters except λ together with Constraint IV.12a implicitly bound λ above and below. Hence, this set of constraints bounds the objective function above and below. Of the nine parameters, λ was chosen to be left unconstrained for two reasons. First, since λ is one of the parameters of greatest interest in the LSCC framework, it is desirable not to require the specification of additional bounds on λ which are unverifiable. Second, this permitted quick calculation of initial feasible solutions of IV.12 by setting the remaining eight parameters at various of their bounds and solving Constraint IV.12a, which was linear only in λ, directly for λ.

[4]Since we are generally more concerned with the variance taking on very large values than very small values, for a given value of MMLSC′, the upper bounds on the individual parameters are viewed as somewhat more important than the lower bounds. However, lower bounds have been included to permit analysis of properties of a well defined (closed) range of variances.

value in his proposal. If K > RT (remedy threshold), they can be interpreted as the Government's best engineering assessment of the regions in which the parameters actually lie for the purpose of calculating statistical risk to the government. In both instances, the LSCC user need not be critically concerned with their accuracy since for a given value of K, Constraint IV.12a limits the patterns in which the nine unknown parameters can vary over their respective domains. For example, if the upper bound of one parameter is increased from, say a to b, the parameter cannot take on values in the interval [a, b], without driving some subset of the remaining parameters to smaller values, (in order to satisfy IV.12a.), thereby largely offsetting any increase in the variance (objective function) that the increased value of the first parameter would have. In other words, Constraint IV.12a dulls the effect of changes in single parameter ranges on the size of the range of allowable variance values.

Since an increase in MMLSC′ reflects an increase in one or more of the nine hardware logistics parameters, some argument can be made for defining the parameter bounds to be (possibly increasing) functions of K. However, this has not yet been done because a concise and workable framework for interpreting LSCC statistical risks that specifically requires these bounds to be defined for all values of K on the domain of β(MMLSC′) has not been found.[5] If bounds were, indeed, defined with respect to all values of K on this domain, it would be useful to explore the impact of alternative bound definitions on such properties as continuity, monotonicity, etc., of the maximum and minimum values of β(MMLSC′) over its domain and resulting impact on the usefulness of the LSCC.

---

[5]Research results bearing on this conclusion will be presented in the last section of Chapter V.

PROPERTIES OF THE NLP FRAMEWORK: ANALYTICAL AND COMPUTATIONAL RESULTS

Problem IV.12 has certain notable mathematical properties. Constraints IV.12c form a hypercube in 8-dimensional space (say S_8) and IV.12b is a hyperplane passing through the hypercube in the first three of its dimensions (say S_3). IV.12b and IV.12c together clearly form a convex constraint region. IV.12a is a smooth surface in nine dimensions (S_9) which intersects the convex region formed by IV.12b and c in S_8. It has no obvious convexity or concavity properties (its Hessian matrix is neither negative nor positive semidefinite (see references 14, p. 89 and 28, p. 30)) and hence the total constraint set in IV.12 is not necessarily convex. Furthermore, the objective (variance) function has no obvious convexity or concavity properties.[6]

Since the above properties provide no guarantees that a global minimum or maximum solution to IV.12 can be found, IV.12 was solved with two different NLP algorithms to determine empirically, if not analytically, how the objective function and constraint region behaved. The first algorithm used was Rosen's Gradient Projection Technique (as coded on the Air Force Logistics Command's CREATE System). While this algorithm led to some useful insights, it is generally regarded as inefficient when nonlinear constraints are present (see reference 8, pp. 315,329, and references 21 and 22) because in this case, an entire inverse matrix of order considerably larger than the total number

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[6] Another alternative here is to substitute $p_1 = 1-p_2-p_3$ throughout IV.12 and delete IV.12b. and IV.12.c1. This makes the constraint set a hypercube in seven dimensions with only the smooth surface, IV.12a, passing through it but diminishes some of the attractive features of IV.12a and the objective function. One can also solve IV.12a for λ and substitute the result in the objective function. In this way, IV.12a is deleted, the entire problem is reduced to seven dimensions, and the remaining constraint set is convex, but the objective function becomes extremely messy. It would be useful to explore these alternative formulations in more depth.

of constraints must be computed at each iteration. The second algorithm used was the Sequential Unconstrained Minimization Technique (SUMT) (see reference 16). This solves the main problem by solving a sequence of unconstrained minimization problems and using a mixed interior-exterior penalty function approach. It is generally more efficient than the first algorithm and was applied to IV.12 with the hope that its quick convergence capability would also permit its application to the general problem in which a base spares estimator is added to MLSC′ (see Chapter V).

The computational approach taken was to start each algorithm at a large number of the extreme points of the convex set in S_8 formed by the intersection of IV.12b and c (henceforth called CS) and to observe the point to which the algorithm converged in each case.[7]

Maximization Problem Results

Rosen's algorithm always converged to one of two alternate apparently global optima from all starting points. Furthermore, these optima were both extreme points of CS. Under the same constraints, the SUMT algorithm virtually always converged to a given point on the edge of CS connecting these two optima and having the same objective function value. However, for a small number of starting points, it converged to another extreme point of CS reflecting a somewhat smaller valued maximum. When the constraints of CS were generalized somewhat from those used for Rosen's algorithm, SUMT converged to the same apparently global optimum, again an extreme point of CS, from all starting points.

[7]Since the computations described below reflect an effort to explore the mathematical properties of the NLP framework in IV.12, the author did not endeavor to use completely representative values for all constants. Hence, numerical results obtained are not entirely meaningful in terms of maximum or minimum values of β(MMLSC′).

Hence, we have the result that both algorithms converge to a small number of solutions, all essentially extreme points of CS in the subspace S_8, from a large number of starting points. This suggests that in the maximization case, the objective function of IV.12 apparently behaves much like a convex function and the entire constraint set of IV.12 behaves much as if it were convex (see reference 8, p.91).[8] This also suggests that a potentially reliable technique for finding a global maximum to IV.12 would be enumeration of objective function values at the extreme points of CS (with λ set equal to that value which satisfies IV.12a in each case).[9]

Minimization Problem Results

In the minimization case, Rosen's algorithm always converged to one of a small number of local minima, starting from a large number of extreme points of CS. Also, certain of the μ_i, $i=1,2,\ldots,5$, coordinates of the minima were not at their bounds and hence, these minima were not extreme points of CS. Under the same constraints, SUMT virtually always converged to the solution point reflecting the smallest objective function value of the set of local minima found by Rosen's algorithm. Hence, it appears that SUMT is more likely than Rosen's algorithm to converge to a global minimum. When the constraints were generalized somewhat, SUMT again always converged to the same apparently global minimum. However, this minimum was always an extreme point of CS. Hence, in this case, the objective function was behaving more as if it were concave.

[8] Although it is possible that the projection of the entire constraint set of IV.12 in S_8 may, indeed, be convex, no proof of this has been found to date.

[9] Further research into the application of partial enumeration techniques to isolate the global maximum would be most useful.

Few firm conclusions can be drawn about the behavior of IV.12 in the minimization case. There is some evidence that SUMT can be depended upon to converge to a global minimum. However, this minimum may not be an extreme point of CS and so enumeration techniques would appear to be of questionable value here.

AN ALTERNATIVE APPROACH: TRANSFORMING THE NLP FRAMEWORK

In the maximization case, Problem IV.12 can be transformed to a problem having a convex objective function and convex constraint set, via an exponential transformation of variables.[10] While this transformed problem was not explored, it is a useful vehicle for further research. It has the property that the global maximum is at an extreme point (see reference 8, p. 91), and initial feasible solutions can be generated easily. Rosen's algorithm, SUMT, or another technique such as the convex simplex algorithm could be usefully applied to this framework.

A STRUCTURE FOR THE MULTIPLE ITEM CASE

Several LSCC applications to date have involved a target LSC for a group of equipment items (see reference 4, p. 3). The approach taken here is to compute a target LSC for each item in terms of a CMF such as that in Figure 1 and to sum these targets to an aggregate target which becomes the formal basis for the LSCC. The mathematical statistics of this aggregate case largely resembles that of the single item case but is more complex in certain respects. The normality assumption is again made, this time with even more justification than in the single item case, because the aggregate estimator is now the sum of statistically independent cost estimates, each of a form similar to IV.1.

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[10] The author is grateful to Prof. Saul Gass of the University of Maryland for suggesting this framework.

It is useful to define some notation for this case and to develop its equivalent to framework IV.12. Let the following quantities be defined (for this subsection only):

a. N = the total number of equipment items for which the target LSC is defined

b. $MLSC'_i$ = the measured LSC for equipment item i (defined in terms similar to the CMF in Figure 1 less its base spares term), $i = 1, 2, \ldots, N$

c. $MLSC'$ = the total measured LSC (the aggregate estimator)

$$= \sum_{i=1}^{N} MLSC'_i$$

d. $MMLSC'_i \equiv E(MLSC'_i)$, $i = 1, 2, \ldots, N$

e. $MMLSC' = E(MLSC') = E(\sum_{i=1}^{N} MLSC'_i) = \sum_{i=1}^{N} MMLSC'_i$

f. $TLSC'_i$ = the target LSC for equipment item i (defined in terms of the Figure 1 CMF less its base spares term), $i = 1, 2, \ldots, N$

g. $TLSC'$ = the aggregate target LSC

$$= \sum_{i=1}^{N} TLSC'_i$$

h. $\sigma^2_{MMLSC'}$ = variance of the MLSC' given its underlying unknown mean is MMLSC'

$$= Var_{MMLSC'} MLSC' = Var_{MMLSC'}\left(\sum_{i=1}^{N} MLSC'_i\right)$$

$$= \sum_{i=1}^{N} Var_{MMLSC'_i} MLSC'_i \Bigg|_{\sum_{i=1}^{N} MMLSC'_i = MMLSC'}$$

IV.13

With these definitions, the power function $\beta(MMLSC')$, has the same form (III.16) as in the single item case,

$$\beta(MMLSC') = Pr_{MMLSC'}\left(Z > \frac{RT-MMLSC'}{\sqrt{\sigma^2_{MMLSC'}}}\right) \text{ where } Z \sim N(0,1). \qquad IV.14$$

However, MMLSC$'$ is a <u>sum</u> of unknown underlying means of the form of IV.9 and $\sigma^2_{MMLSC'}$ is a sum of unknown underlying variances of the form of IV. 10. Furthermore, $\sigma^2_{MMLSC'}$ is now nonunique in <u>two</u> respects. First, for a given item i and a given unknown mean value $MMLSC'_i$,

$$\mathrm{Var}_{MMLSC'_i}(MLSC'_i)$$

is nonunique. Secondly, for a given MMLSC$'$ value, there exist an infinite number of combinations of the $MMLSC'_i$'s, i = 1,2,...,N, such that

$$\sum_{i=1}^{N} MMLSC'_i = MMLSC',$$

each with its own range of values of $\mathrm{Var}_{MMLSC'_i}(MLSC'_i)$.

In view of these properties, it is useful to formulate an analog to Problem IV.12 for the multiple item case, again to explore the range of unknown variances, $\sigma^2_{MMLSC'}$ for a given value of the unknown mean, MMLSC$'$. The framework is defined as follows:

$$\left.\begin{array}{ll}
\text{maximize(minimize)} & \sigma^2_{MMLSC'} = \sum_{i=1}^{N} \mathrm{Var}_{MMLSC'_i} MLSC'_i \Big|_{\sum_{i=1}^{N} MMLSC'_i = MMLSC'} \\
\text{subject to} & \text{a. } MMLSC' = \sum_{i=1}^{N} MMLSC'_i = K \\
 & \text{b. } p_{i1} + p_{i2} + p_{i3} = 1, \quad i=1,2,..,N \\
 & \text{c. } LB_{ij} \le p_{ij} \le UB_{ij}, \quad j=1,2,3;\ i=1,2,...,N \\
 & \quad\ LB_{i,j+3} \le \mu_{ij} \le UB_{i,j+3}, \quad j=1,2,3,4,5; \\
 & \qquad\qquad\qquad\qquad\qquad i=1,2,..,N \\
 & \quad\ (LB_{ij} \ge 0,\ j=1,2,...,8;\ i=1,2,..,N) \\
 & \text{d. } LB_{i9} \le MMLSC'_i \le UB_{i9}, \quad i=1,2,..,N
\end{array}\right] \quad \text{IV.15}$$

This framework is essentially an N-fold multiplicity version of IV.12 except for Constraints IV.15d. These are defined to confine the unknown underlying mean, $MMLSC'_i$, for each item to a meaningful range about its target while the unknown underlying aggregate mean is being held at the value, K. For example, the upper and lower bounds, UB_{i9} and LB_{i9}, on $MMLSC'_i$ can be defined as the endpoints of the domain of interest of the power function for each item, adjusted to be consistent with the value of K to which the aggregate mean is constrained.[11]

The existence of statistical independence between MLSC estimates for different items should tend to result in a range of smaller variances in IV.15 than in the single item case.[12] It would be most useful to attempt to verify this reduced variance phenomenon as well as to explore other properties of the framework. Problem IV.15 can be solved to determine the behavior of the maximum (minimum) of $\sigma^2_{MMLSC'}$ as a function of different values of K using dynamic programming or generalized Lagrange multipliers. The Lagrangian approach would solve a series of single item problems.

---

[11]The bounds, LB_{i9} and UB_{i9}, i=1,2,...,N, largely depend on the values of the power function, $\beta(MMLSC'_i)$ at various points in the domain of the target $TLSC'_i$. As $\beta(MMLSC'_i)$ flattens out, the domain of interest, $[LB_{i9}, UB_{i9}]$ tends to lengthen. Hence, it may be necessary to solve IV.12 for several values of K for each equipment item to determine appropriate values of LB_{i9}, UB_{i9}, i=1,2,...,N, before solving IV.15.

[12]Other evidence of this reduced variance property is described in Chapter VI.

V. MEAN AND VARIANCE PROPERTIES OF MLSC

Recall that the base spares cost term in the CMF of Figure 1 is defined to provide a safety stock of base spares sufficiently large to guarantee an appropriately low number of expected backorders at each base at any point in time. The inclusion of this safety stock considerably increases the CMF's computational complexity. In addition, the mathematics of the estimator of the resulting cost term is much more detailed than that of the terms of MLSC' and its mean, variance, and cross-product moments with various other terms of MLSC are not easily computed.

The goals of this chapter are to develop the estimator of the Figure 1 base spares cost term used in recent LSCC applications, to examine its mathematical properties, and to examine the impact of these properties on the mean and variance properties of the MLSC. It is appropriate to begin by looking at the mathematics underlying the formulation of the Figure 1 base spares cost term, itself.

THE EXPECTED BACKORDER FUNCTION AND THE BASE SPARES COST TERM

From Figure 1, the number of expected backorders, $XBO(i,\theta t)$ at a given base when the stock level is i and the Poisson arrival rate is θt is defined as

$$\sum_{x>i} (x-i)\, p(x \mid \theta t), \quad i \geqq 0, \ \theta t > 0. \qquad \text{V.1}$$

Letting $\gamma \equiv \theta t$ for simplicity of exposition, we have

$$XBO(i,\gamma) = \sum_{x>i} (x-i)\, p(x \mid \gamma), \quad i \geqq 0, \ \gamma > 0. \qquad \text{V.2}$$

This function is well behaved in many respects. First, it is easily shown (see Lemma B1) that

$$XBO(i+1,\gamma) - XBO(i,\gamma) = \sum_{x=0}^{i} p(x \mid \gamma) - 1 \quad < 0 \qquad \text{V.3}$$

and hence,

$$XBO(i,\gamma) > XBO(i+1,\gamma), \; i \geqq 0, \; \gamma > 0, \qquad \text{V.4}$$

or XBO(i,γ) is a decreasing function of i for a given $\gamma > 0$ with differences, XBO(i,γ) - XBO(i+1, γ), decreasing as i increases. For example,

$$\left.\begin{aligned} XBO(0,\gamma) &= \gamma \\ XBO(1,\gamma) &= \gamma + \underbrace{(e^{-\gamma} - 1)}_{< 0} \\ XBO(2,\gamma) &= \gamma + \underbrace{(e^{-\gamma} - 1)}_{< 0} + \underbrace{(e^{-\gamma} + \gamma e^{-\gamma} - 1)}_{< 0} \end{aligned}\right] \qquad \text{V.5}$$

and so on. V.5 illustrates the fact that with no stock (i = 0), the expected number of backorders is simply γ, the mean number of arrivals of failed items. Furthermore, as the stock level, i, increases, the expected number of backorders is reduced by successively smaller amounts. Using the format if V.5, we also can rewrite V.2 as

$$XBO(i,\gamma) = \gamma + \sum_{s=0}^{i-1} \left(\sum_{x=0}^{s} p(x \mid \gamma) - 1\right), \; i \geqq 1, \; \gamma > 0. \qquad \text{V.6}$$

The backorder function is also well behaved with respect to γ. It is easily shown (see Lemma B2) that

$$\frac{\partial\, XBO(i,\gamma)}{\partial\gamma} = \sum_{x=i}^{\infty} p(x \mid \gamma), \; i \geqq 0, \; \gamma > 0, \qquad \text{V.7}$$

and

$$\frac{\partial^2 XBO(i,\gamma)}{\partial\gamma^2} = p(i-1 \mid \gamma), \; i \geqq 1, \; \gamma > 0. \qquad \text{V.8}$$

Hence, XBO(i,γ) is an increasing continuous convex function (strictly convex for $i \geqq 1$) of γ with

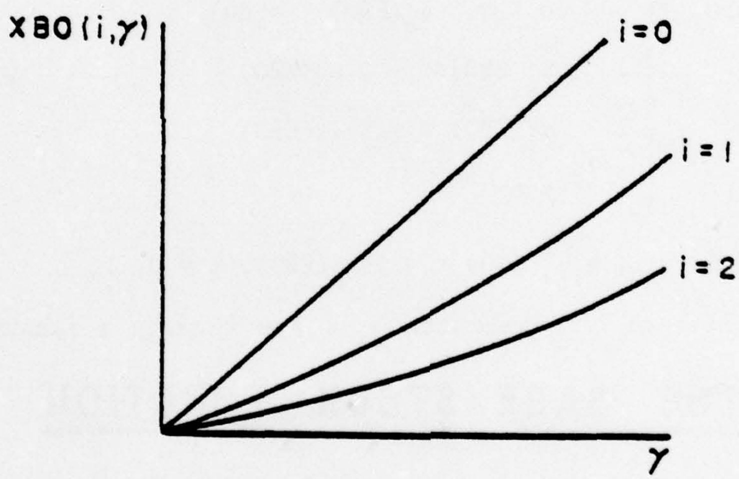
$$0 < \frac{\partial \text{ XBO}(i,\gamma)}{\partial\gamma} \leqq 1, \ i \geqq 0, \ \gamma > 0, \qquad \text{V.9}$$

and

$$0 < \frac{\partial^2 \text{ XBO } (i,\gamma)}{\partial\gamma 2} < 1, \ i > 1, \ \gamma > 0. \qquad \text{V.10}$$

Taking note of these properties and in particular, the facts that ∂ XBO $(0,\gamma)/\partial\gamma = 1$ and ∂^2 XBO$(0,\gamma)/\partial\gamma^2 = 0$, the graphs of XBO$(i,\gamma)$ as a function of γ for $i = 0, 1, 2, \ldots$ are as shown in Figure 3,

Figure 3. **THE BACKORDER FUNCTION**



i.e., XBO$(0,\gamma)$ has a slope of 1 while the slopes for $i > 0$ are less but all approach 1 asymptotically.

The base stock level, STK(γ, EBO) in the Figure 1 CMF is defined to be that minimum value of i such that XBO$(i,\gamma) \leqq$ EBO, where EBO is a prespecified threshold, i.e.,

$$\text{STK}(\gamma, \text{EBO}) = \begin{cases} 0, & \text{XBO}(0,\gamma) \leqq \text{EBO (i.e., } \gamma \leqq \text{EBO}), \\ 1, & \text{XBO}(0,\gamma) > \text{EBO but XBO}(1,\gamma) \leqq \text{EBO} \\ & \text{(i.e., } \gamma > \text{EBO but } \gamma + (e^{-\gamma} - 1) \leqq \text{EBO}), \end{cases} \qquad \text{V.11}$$

$$\left\lfloor \begin{array}{l} 2,\ XBO(1,\gamma) > EBO \text{ but } XBO(2,\gamma) \leqq EBO \\ \quad (\text{i.e., } \gamma + (e^{-\gamma} - 1) > EBO \text{ but } \gamma + (e^{-\gamma} - 1) + \\ \quad (e^{-\gamma} + \gamma e^{-\gamma} - 1) \leqq EBO), \\ \text{etc.} \end{array} \right.$$

where Definition V.11 relies on the fact that XBO(i,γ) as a decreasing function of i. Furthermore, since XBO(i,γ) is an increasing function of γ, the largest value of γ such that STK(γ, EBO) = i is that γ such that XBO(i,γ) = EBO. Accordingly, let

$$a_i(EBO) \equiv \text{the largest value of } \gamma \text{ such that } STK(\gamma, EBO) = i, \quad i = 0, 1, 2, \ldots \qquad \text{V.12}$$

By these definitions, $EBO = XBO(0,a_o(EBO)) = XBO(1,a_1(EBO)) = XBO(2,a_2(EBO)) = \ldots$ Also, the base stock level can now be defined as

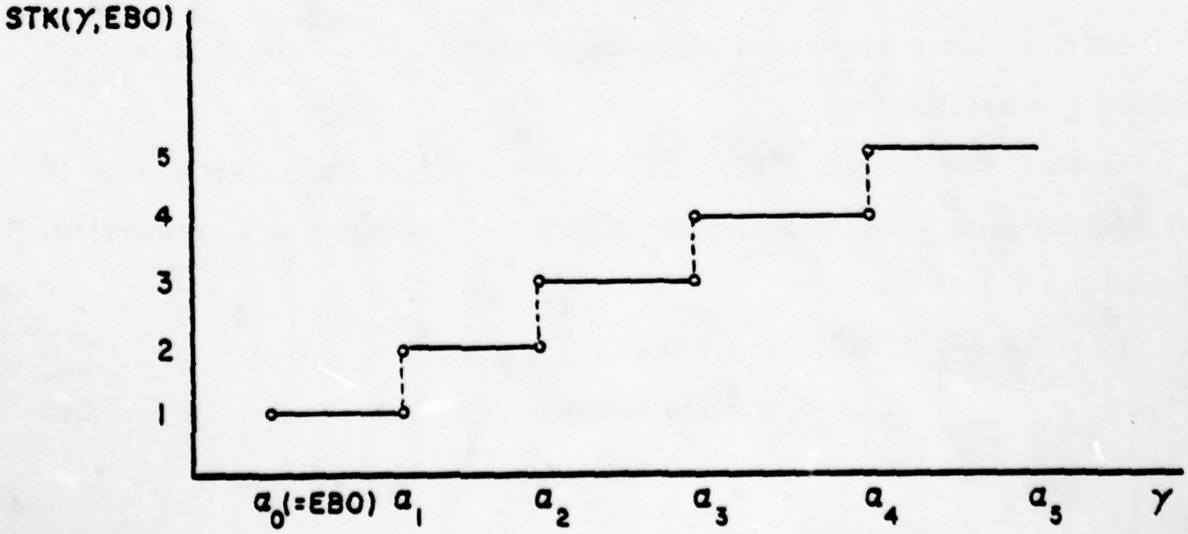
$$STK(\gamma, EBO) = \begin{cases} 0 & 0 < \gamma \leqq a_o(EBO) \quad (=EBO) \\ 1 & a_o(EBO) < \gamma \leqq a_1(EBO) \\ 2 & a_1(EBO) < \gamma \leqq a_2(EBO) \\ & \text{etc.} \end{cases} \qquad \text{V.13}$$

or

$$STK(\gamma, EBO) = i,\ a_{i-1}(EBO) < \gamma \leqq a_i(EBO),\ i = 0, 1, \ldots, \qquad \text{V.14}$$

as illustrated in Figure 4. While the a_i's are clearly a function of EBO,

Figure 4. **THE BASE STOCK FUNCTION**



their successive differences can be shown to be less than 1 ($a_i - a_{i-1} < 1$, $i = 1, 2, \ldots$ (see Lemma B3) and these differences approach 1 as i increases. Some typical values of the a_i's and successive differences appear in Figure 5. These various properties of the a_i's will be necessary to prove convergence of some of the moments of the estimator of STK(γ, EBO). We now proceed to develop this estimator.[1]

FIGURE 5. SAMPLE VALUES OF THE a_i(EBO)'s

For EBO = .01		For EBO = .10	
$a_o = .01$	$a_o - 0 = .01$	$a_o = .10$	$a_o - 0 = .10$
$a_1 = .13$	$a_1 - a_o = .12$	$a_1 = .48$	$a_1 - a_o = .38$
$a_2 = .40$	$a_2 - a_1 = .27$	$a_2 = .98$	$a_2 - a_1 = .50$
$a_3 = .76$	$a_3 - a_2 = .36$	$a_3 = 1.55$	$a_3 - a_2 = .57$
$a_4 = 1.18$	$a_4 - a_3 = .42$	$a_4 = 2.15$	$a_4 - a_3 = .60$
$a_5 = 1.65$	$a_5 - a_4 = .47$	$a_5 = 2.79$	$a_5 - a_4 = .64$
$a_6 = 2.15$	$a_6 - a_5 = .50$	$a_6 = 3.45$	$a_6 - a_5 = .66$

THE BASE SPARES ESTIMATOR AND ASSOCIATED MOMENTS

The estimator of the base spares term used in most recent LSCC applications has a form identical to cost term itself except that the argument of the function, $\gamma = \theta t$, is replaced by its estimator, $\tilde{\gamma} \equiv \widetilde{\theta t}$, as defined in II.5, i.e.,

$$\widetilde{STK}(\tilde{\gamma}, EBO) = i, \quad a_{i-1}(EBO) < \tilde{\gamma} \leq a_i(EBO), \quad i = 0, 1, \ldots \qquad \text{V.15}$$

By grouping constants according to III.5c and substituting appropriate estimators from III.4 into the II.5 expression for $\tilde{\gamma} = \widetilde{\theta t}$, it simplifies to

$$\tilde{\gamma} = \widetilde{\theta t} = K_6 X_2(T) + K_7 X_3(T). \qquad \text{V.16}$$

[1]Given the definition of the a_i(EBO)'s (V.12) and the facts that XBO (i,γ) is a monotonically increasing function of γ, $i \geqq 1$, and a_{i+1}(EBO) $- a_i$(EBO) < 1, $i \geqq 0$, the a_i(EBO)'s can be computed easily via a sequential binary search (Bolzano's Method; see reference 27, pp. 230, 231).

Making this substitution in V.15 yields the following expression for the base spares estimator as a function of the two observed variables, $X_2(T)$ and $X_3(T)$ and the exogenously determined threshold, EBO:

$$\widetilde{STK}(\tilde{\gamma}, EBO) = i, \; a_{i-1}(EBO) < K_6 X_2(T) + K_7 X_3(T) \leq a_i(EBO), \qquad V.17$$

$$i = 0, 1, \ldots$$

This is simply another way of writing III.5b.

By the definition of the n'th moment of a random variable, we have

$$E(\widetilde{STK}^n(\tilde{\gamma}, EBO)) = \sum_{i=0}^{\infty} i^n \Pr(a_{i-1}(EBO) < \tilde{\gamma} \leq a_i(EBO)). \qquad V.18$$

Substituting V.16 in V.18 leads to (see Lemma B4) the following expression, henceforth called Pr(i, EBO), for the probability that the estimator, $\tilde{\gamma}$, will take on a value on the domain of the i'th step of $\widetilde{STK}(\tilde{\gamma}, EBO)$:

$$\Pr(i, EBO) \equiv \sum_{\substack{m_2, m_3, \\ a_{i-1}(EBO) < K_6 m_2 + K_7 m_3 \leq a_i(EBO)}} \left[\frac{(p_2 \lambda T)^{m_2}}{m_2!} e^{-p_2 \lambda T}\right] \left[\frac{(p_3 \lambda T)^{m_3}}{m_3!} e^{-p_3 \lambda T}\right] \qquad V.19$$

Hence the mean and variance of the base spares term can be expressed as follows:

$$E(\widetilde{STK}(\tilde{\gamma}, EBO)) = \sum_{i=0}^{\infty} i \Pr(i, EBO) \qquad V.20$$

$$Var(\widetilde{STK}(\tilde{\gamma}, EBO)) = \sum_{i=0}^{\infty} i^2 \Pr(i, EBO) - [\, E(\widetilde{STK}(\tilde{\gamma}, EBO))]^2 \qquad V.21$$

To develop an expression for the variance of the entire MLSC (see III.11), expressions for product moments involving $\widetilde{STK}(\tilde{\gamma}, EBO)$ and those terms of MLSC' (see IV.1) which are functions of $X_2(T)$ or $X_3(T)$ must also be developed. The product moment involving $\widetilde{STK}(\tilde{\gamma}, EBO)$ and the failure estimate $X_3(T)$ in the depot spares cost estimator (see the first term of IV.1) has the form,

$$E(\widetilde{STK}(\tilde{\gamma}, EBO)\, X_3(T)) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} ij \Pr(\widetilde{STK}(\tilde{\gamma}, EBO) = i, X_3(T) = j). \qquad V.22$$

By Lemma B5, the joint probability in V.22 is expressed as follows:

$$\Pr(\widetilde{STK}(\tilde{\gamma}, EBO)=i,\ X_3(T)=j) = \frac{(p_3\lambda T)^j}{j!} e^{-p_3\lambda T} \left[\sum_{m} \frac{(p_2\lambda T)^m}{m!} e^{-p_2\lambda T}\right], \qquad \text{V.23}$$

$$\max\left(0, \frac{a_{i-1}(EBO)-K_7 j}{K_6}\right) < m \le \left(\frac{a_i(EBO)K_7 j}{K_6}\right)$$

$$i \ge 0,\ 0 \le j \le M_i^1 \equiv [a_i(EBO)/K_7]$$

where the values that $X_3(T)$ can assume are bounded above by the fact that $\tilde{\gamma} = K_6\ X_2(T) + K_7\ X_3(T)$ is confined to the domain of the i'th step of $\widetilde{STK}(\tilde{\gamma}$, EBO). Substituting V.23 in V.22 leads to the following expression for the product moment:

$$E(\widetilde{STK}(\tilde{\gamma}, EBO)X_3(T)) = \sum_{i=0}^{\infty} i \sum_{j=0}^{M_i^1} j\ \frac{(p_3\lambda T)^j}{j!} e^{-p_3\lambda T}\left[\sum_{m} \frac{(p_2\lambda T)^m}{m!} e^{-p_2\lambda T}\right], \qquad \text{V.24}$$

$$\max\left(0, \frac{a_{i-1}(EBO)-K_7 j}{K_6}\right) < m \le \frac{a_i(EBO)-K_7 j}{K_6}$$

$$(M_i^1 = [a_i(EBO)/K_7])$$

Furthermore, by symmetry,

$$E(\widetilde{STK}(\tilde{\gamma}, EBO)X_2(T)) = \sum_{i=0}^{\infty} i \sum_{j=0}^{M_i^2} j\ \frac{(p_2\lambda T)^j}{j!} e^{-p_2\lambda T}\left[\sum_{m} \frac{(p_3\lambda T)^m}{m!} e^{-p_3\lambda T}\right], \qquad \text{V.25}$$

$$\max\left(0, \frac{a_{i-1}(EBO)-K_6 j}{K_7}\right) < m \le \frac{a_i(EBO)-K_6 j}{K_7}$$

$$(M_i^2 = [a_i(EBO)/K_6])$$

The other product moments of interest all involve $\widetilde{STK}(\tilde{\gamma}$, EBO) and a compound Poisson sum having an upper limit of either $X_2(T)$ or $X_3(T)$. The

general form of these product moments is illustrated as follows by that one involving $S\widetilde{T}K(\tilde{\gamma}, EBO)$ and the sum,

$$\sum_{k=0}^{X_2(T)} Y_{4k}$$

(from the sixth estimator term of MLSC' (IV.1)):

$$E\left(\widetilde{STK}(\widetilde{\gamma},EBO)\sum_{k=0}^{X_2(T)} Y_{4k}\right) = \mu_4 \sum_{i=0}^{\infty} i \sum_{j=0}^{M_1^2} j \frac{(p_2\lambda T)^j}{j!} e^{-p_2\lambda T}\left[\sum_{m} \frac{(p_3\lambda T)^m}{m!} e^{-p_3\lambda T}\right] \quad \text{V.26}$$

$$\max\left(0, \frac{a_{i-1}(EBO)-K_6 j}{K_7}\right) < m \leq \frac{a_i(EBO)-K_6 j}{K_7}$$

$$(M_1^2 = [a_i(EBO)/K_6])$$

$$= \mu_4 \, E(\widetilde{STK}(\widetilde{\gamma},EBO)X_2(T)) \quad \text{V.27}$$

In other words, by Expression V.26 (developed in Lemma B6), we have the result that the product moment involving the estimator $S\widetilde{T}K(\tilde{\gamma}, EBO)$ and a Compound Poisson sum estimator in MLSC' is simply equal to the mean of any one of the i.i.d. elements of the sum times the product moment involving $S\widetilde{T}K(\tilde{\gamma}, EBO)$ and the limit of the sum. By symmetry, the other relevant product moments of this form (refer to IV.1 and IV.3) are:

$$\left.\begin{aligned}
E(S\widetilde{T}K(\tilde{\gamma}, EBO) \sum_{k=1}^{X_2(T)} Y_{1k}^{(2)}) &= \mu_1 E(S\widetilde{T}K(\tilde{\gamma}, EBO)\ X_2(T)) \\
E(S\widetilde{T}K(\tilde{\gamma}, EBO) \sum_{k=1}^{X_2(T)} Y_{3k}^{(2)}) &= \mu_3 E(S\widetilde{T}K(\tilde{\gamma}, EBO)\ X_2(T)) \\
E(S\widetilde{T}K(\tilde{\gamma}, EBO) \sum_{k=1}^{X_3(T)} Y_{1k}^{(3)}) &= \mu_1 E(S\widetilde{T}K(\tilde{\gamma}, EBO)\ X_3(T))
\end{aligned}\right] \quad \text{V.28}$$

$$E(\widetilde{STK}(\tilde{\gamma}, EBO) \sum_{k=1}^{X_3(T)} Y_{3k}^{(3)}) = \mu_3 E(\widetilde{STK}(\tilde{\gamma}, EBO) X_3(T))$$

$$E(\widetilde{STK}(\tilde{\gamma}, EBO) \sum_{k=1}^{X_3(T)} Y_{5k}) = \mu_5 E(\widetilde{STK}(\tilde{\gamma}, EBO) X_3(T)) \quad \text{V.28 (cont.)}$$

For convenience, we also define those covariances involving $\widetilde{STK}(\tilde{\gamma}, EBO)$ and either of the Poisson random variables, $X_2(T)$ or $X_3(T)$, at this time. By IV.5, V.20, and V.25, we have

$$\begin{aligned}
Cov(\widetilde{STK}(\tilde{\gamma}, EBO), X_2(T)) &\equiv E(\widetilde{STK}(\tilde{\gamma}, EBO) X_2(T)) \\
&\quad - E(\widetilde{STK}(\tilde{\gamma}, EBO)) E(X_2(T)) \\
&= \sum_{i=0}^{\infty} i \sum_{j=0}^{M_i^2} j \frac{(p_2\lambda T)^j}{j!} e^{-p_2\lambda T} \left[\sum_{\substack{m \\ \max\left(0, \frac{a_{i-1}(EBO)-K_6 j}{K_7}\right) < m \le \frac{a_i(EBO)-K_6 j}{K_7}}} \frac{(p_3\lambda T)^m}{m!} e^{-p_3\lambda T} \right] \\
&\quad - (p_3\lambda T) \sum_{i=0}^{\infty} i \sum_{\substack{m_2, m_3 \\ a_{i-1}(EBO) < K_6 m_2 + K_7 m_3 \le a_i(EBO)}} \left[\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T} \right] \left[\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T} \right] \\
&\quad (M_i^2 = [a_i(EBO)/K_6])
\end{aligned} \quad \text{V.29}$$

and by IV.5, V.20, and V.24,

$$\begin{aligned}
\widetilde{Cov}(\widetilde{STK}(\tilde{\gamma}, EBO), X_3(T) &\equiv E(\widetilde{STK}(\tilde{\gamma}, EBO) X_3(T)) \\
&\quad - E(\widetilde{STK}(\tilde{\gamma}, EBO)) E(X_3(T)) \\
&= \sum_{i=0}^{\infty} i \sum_{j=0}^{M_i^1} j \frac{(p_3\lambda T)^j}{j!} e^{-p_3\lambda T} \left[\sum_{\substack{m \\ \max\left(0, \frac{a_{i-1}(EBO)-K_7 j}{K_6}\right) < m \le \frac{a_i(EBO)-K_7 j}{K_6}}} \frac{(p_2\lambda T)^m}{m!} e^{-p_2\lambda T} \right] \\
&\quad - (p_3\lambda T) \sum_{i=0}^{\infty} i \sum_{\substack{m_2, m_3 \\ a_{i-1}(EBO) < K_6 m_2 + K_7 m_3 \le a_i(EBO)}} \left[\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T} \right] \left[\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T} \right] \\
&\quad (M_i^1 = [a_i(EBO)/K_7])
\end{aligned} \quad \text{V.30}$$

These terms will appear in the expression for Var MLSC to be defined precisely in a later section of this chapter.

MOMENT PROPERTIES

The properties of the base spares estimator mean,

$$E(\widetilde{STK}(\tilde{\gamma},EBO)) = \sum_{i=0}^{\infty} i \sum_{\substack{m_2,m_3 \ni \\ a_{i-1}(EBO)<K_6 m_2+K_7 m_3 \leq a_i(EBO)}} \left[\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right]\left[\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right] \qquad \text{V.31}$$

(from V.19 and V.20) are of special importance, particularly because E(MLSC) (to be defined precisely later) is a linear function of V.31.

A most important property of V.31 is that it is finite, i.e. (see Lemma B7),

$$E(\widetilde{STK}(\tilde{\gamma}, EBO)) < \infty.^2 \qquad \text{V.32}$$

Furthermore, for most values of interest of its arguments, p_2, p_3, and λ, it converges fairly rapidly. Figure 6 illustrates this rate of convergence for two different values of λ and typical values for p_2, p_3, and relevant constants. Notice that for $\lambda = .005$ (MTBF = 200 hr.) the convergence is fairly rapid. Given the values of the parameters, $p_2\lambda T$ and $p_3\lambda T$, of $X_2(T)$ and $X_3(T)$, respectively, the cumulative distribution function (cdf) of the random variable, $\tilde{\gamma} = K_6\, X_2(T) + K_7\, X_3(T)$, or alternatively, of $\widetilde{STK}(\tilde{\gamma}, EBO)$, increases rapidly with each step of $\widetilde{STK}(\tilde{\gamma}, EBO)$. There are large lumps of probability corresponding to several steps, particularly 3, 4, and 5, and the cdf of $\tilde{\gamma}$

[2] By Lemma B7, V.21 is also finite and by Lemma B8, V.24 and V.25 are finite. It follows from the finiteness of V.24 and V.25 and the form of V.27 and V.28 that V.27 and V.28 are finite.

Figure 6. THE RATE OF CONVERGENCE OF $E(\widetilde{STK}(\tilde{\gamma}, EBO))$ AS A FUNCTION OF λ

Other Parameter Values:

EBO = .1; a_i (EBO) generated accordingly, i = 0, 1, 2, ...

K_6 = PFFH x QPA x UF x BRCT/(NB* T) = .09774*

K_7 = PFFH x QPA x UF x (OSTCON x (1.0-OS) + OSTOS x OS)/(NB x T) = .33306*

T = 3500 hrs.

p_2 = .95, p_3 = .05

For λ = .005 (MTBF = 200 hrs., $p_2\lambda T$ = 16.625, $p_3\lambda T$ = .875 γ=1.9163, STK = 4):

n	$E_n(\widetilde{STK}(\tilde{\gamma}, EBO)) \equiv \sum_{i=0}^{n} i\,P(i,EBO)$	$Pr(a_{i-1}(EBO) < \tilde{\gamma} \le a_i(EBO)) = Pr(\widetilde{STK}(\tilde{\gamma}, EBO) = i)$	$Pr(\tilde{\gamma} \le a_n(EBO)) = \sum_{i=0}^{n} Pr(a_{i-1}(EBO) < \tilde{\gamma} \le a_i(EBO))$
0	0	.000000439	.000000439
1	.00010301	.000103012	.000103452
2	.05057295	.025234968	.025338420
3	.66734903	.205592025	.230930444
4	2.56082500	.473368994	.704299436
5	3.78368960	.244572919	.948872358
6	4.06829099	.047433565	.996305920
7	4.09321366	.003560339	.999866265
8	4.09426090	.000130943	.999999209
9	4.09428569	.000002754	.999999961

*These values are based on the assumption that PFFH = 18,420, QPA = 1.0, UF = 1.0, BRCT = .13, NB = 7, OSTCON = .40, OS = .43, and OSTOS = .50. Definitions of these parameters appear in II.1 and their values were chosen to be representative of recent LSCC applications.

Figure 6. (cont'd). **THE RATE OF CONVERGENCE OF $E(\widetilde{STK}(\tilde{\gamma},EBO))$ AS A FUNCTION OF λ**

For $\lambda = .0005$ (MTBF = 2000 hrs., $p_2\ \lambda T = 1.6625$, $p_3\ \lambda T = .0875$, $\gamma = .1916$, STK = 1):

n	$E_n(\widetilde{STK}(\tilde{\gamma}, EBO)) \equiv \sum_{i=0}^{n} i\ Pr(i,EBO)$	$Pr(a_{i-1}(EBO) < \tilde{\gamma} \leq a_i(EBO)) = Pr(\widetilde{STK}(\tilde{\gamma}, EBO) = i)$	$Pr(\tilde{\gamma} \leq a_n(EBO)) = \sum_{i=0}^{n} Pr(a_{i-1}(EBO) < \tilde{\gamma} \leq a_i(EBO))$
0	0	.288899180	.288899180
1	.46902514	.469025139	.757924320
2	.60453630	.067755583	.825679904
3	.60617076	.000544818	.826224723
4	.60617610	.000001335	.826226054

levels off rapidly thereafter, reaching .99999961 at a_9(EBO). This rapid increasing and then rapid leveling off of the cdf causes the n'th partial sum, $E_n(\widetilde{STK}(\tilde{\gamma}, EBO))$ to behave similarly. Accordingly, the small amount of probability remaining to the right of a_9(EBO) together with the proof of finiteness of $E(\widetilde{STK}(\tilde{\gamma}, EBO))$ (Lemma B7) justify $E_9(\widetilde{STK}(\tilde{\gamma}, EBO))$ as a reasonable approximation of the infinite series in a computational environment.

This rapid convergence will occur for typical values of p_2 and p_3 and for most values of λ of interest. However, for low values of λ (very high MTBF), the convergence is slower to the point of causing some computational difficulties. The second example in Figure 6 (λ = .0005 (MTBF = 2000 hrs.)) illustrates this problem. Here, the values of the parameters, $p_2\lambda T$ and $p_3\lambda T$, are considerably smaller than in the first example and cause the probability distribution of $\tilde{\gamma}$ to have a very large right hand tail with very little probability within each step of $\widetilde{STK}(\tilde{\gamma}, EBO)$ beyond i = 3 or 4. Hence, the cdf of $\tilde{\gamma}$ increases very rapidly at first but begins to level off at about .82 so that many more terms of $E(\widetilde{STK}(\tilde{\gamma}, EBO))$ must be computed before finding an n'th partial sum considered to be an adequate approximation of the infinite series.

In most cases, this problem of slow convergence can be circumvented in assessing LSCC statistical risk via minor modifications to the risk assessment framework. This problem will be addressed further in later sections.[3]

Another interesting property of $E(\widetilde{STK}(\tilde{\gamma}, EBO))$ is the nature of its functional dependence on the parameter λ. Figure 7 illustrates this dependence. Except for the first two plotted points which are probably inadequate

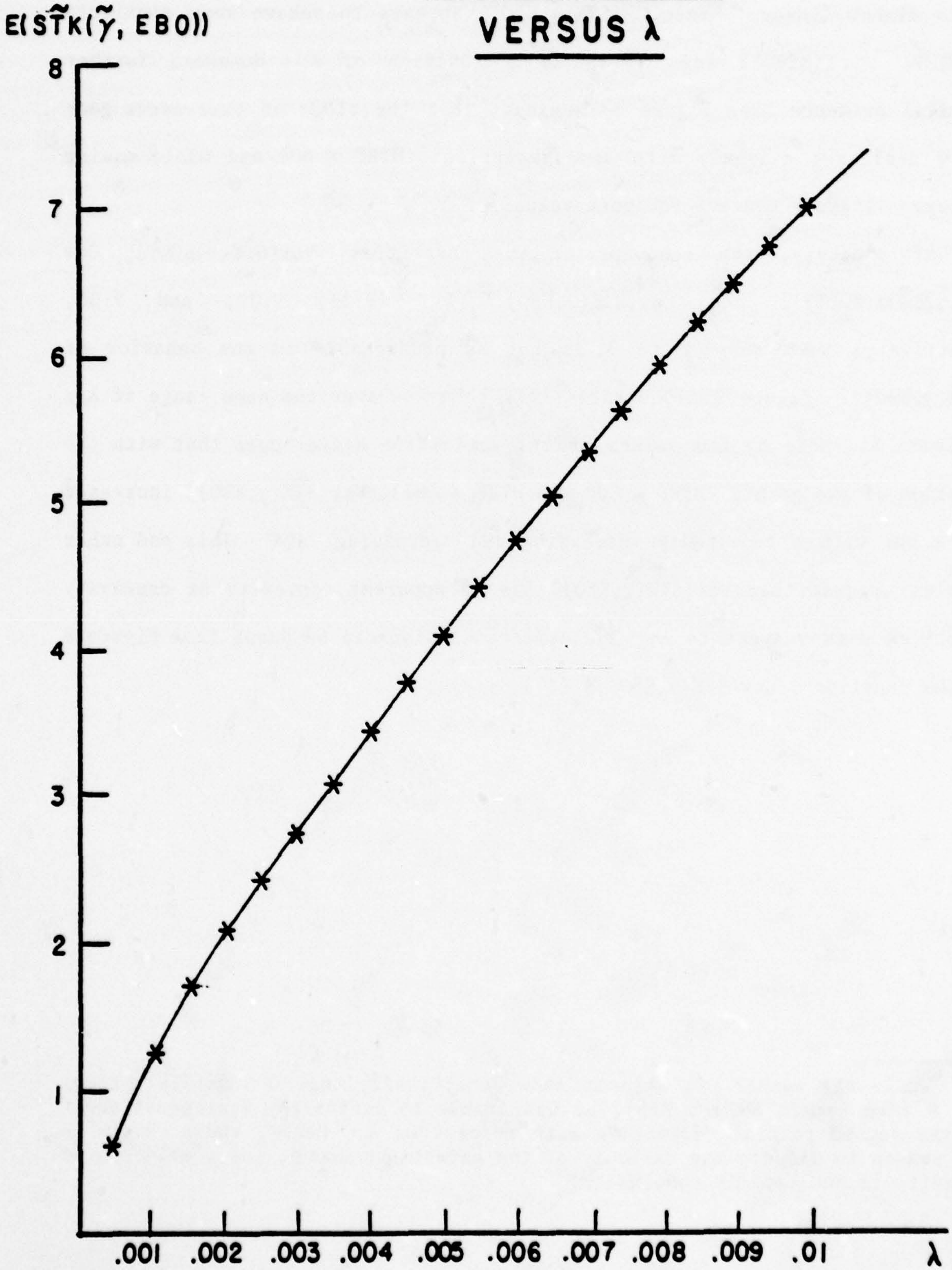
[3]The rate of convergence properties of other moments involving $\widetilde{STK}(\tilde{\gamma}$, EBO) including V.21, V.24, V.27, and V.28 are essentially the same as $E(\widetilde{STK}(\tilde{\gamma}, EBO))$ and hence will not be dealt with explicitly here.

Figure 7. E(S$\tilde{T}$K($\tilde{\gamma}$, EBO)) Versus λ*

λ	MTBF	E_n(S$\tilde{T}$K($\tilde{\gamma}$, EBO))	STK	γ	Pr($\tilde{\gamma} \leq a_n$(EBO))
.0005	2000	.606	1.0	.1916	.82623
.0010	1000	1.225	1.0	.3833	.96980
.0015	667	1.702	2.0	.5749	.99475
.0020	500	2.080	2.0	.7665	.99909
.0025	400	2.419	2.0	.9582	.99984
.0030	333	2.757	3.0	1.1498	.99997
.0035	286	3.101	3.0	1.3414	> .99999
.0040	250	3.442	3.0	1.5331	> .99999
.0045	222	3.774	4.0	1.7247	> .99999
.0050	200	4.094	4.0	1.9163	> .99999
.0055	182	4.404	4.0	2.1080	> .99999
.0060	167	4.707	5.0	2.2996	> .99999
.0065	154	5.006	5.0	2.4912	> .99999
.0070	143	5.304	5.0	2.6829	> .99999
.0075	133	5.600	6.0	2.8745	> .99999
.0080	125	5.895	6.0	3.0661	> .99999
.0085	118	6.188	6.0	2.2578	> .99999
.0090	111	6.478	6.0	3.4494	> .99999
.0095	105	6.766	7.0	3.6410	> .99999
.0100	100	7.051	7.0	3.8327	> .99999

*Values of all parameters and variables not tabulated above are the same as in the examples of Figure 6. In particular, p_2 = .95 and p_3 = .05.

Figure 7 (cont.). $E(S\tilde{T}K(\tilde{\gamma}, EBO))$ VERSUS λ



approximations of $E(\widetilde{STK}(\tilde{\gamma},EBO))$ because of their low λ values, (and hence slow convergence) all plotted points suggest that $E(\widetilde{STK}(\tilde{\gamma},EBO))$ increases with λ and is almost linear. Hence, $E(\widetilde{STK}(\tilde{\gamma},EBO))$ appears to behave very similarly to MMLSC' = E(MLSC') (see IV.9) as a function of λ. However, further empirical evidence (see Figure 8) suggests that the slope of this curve generally decreases slightly with some exceptions (MTBF = 400 and 333), making the curve slightly <u>concave</u> for most values of λ.[4]

Unfortunately, the behavior of the quantities, $Var(\widetilde{STK}(\tilde{\gamma},EBO))$, $Cov(\widetilde{STK}(\tilde{\gamma},EBO),X_2(T))$, and $Cov(\widetilde{STK}(\tilde{\gamma},EBO),X_3(T))$ (V.21, V.29, and V.30, respectively) with respect to λ is not as predictable as the behavior of $E(\widetilde{STK}(\tilde{\gamma},EBO))$. Figure 9 illustrates this behavior over the same range of λ's as Figure 8. Note by the values of its successive differences that with the exception of two points (MTBF = 500 and MTBF = 667), $Var(\widetilde{STK}(\tilde{\gamma},EBO))$ increases with λ but with a repeatedly increasing and decreasing rate. This and other examples suggest that $Var(\widetilde{STK}(\tilde{\gamma},EBO))$ has no apparent convexity or concavity properties with respect to λ. The same conclusion can be drawn from Figure 9 for the functions, $Cov(\widetilde{STK}(\tilde{\gamma},EBO),X_i(T))$, $i = 2, 3$.

[4]While the author was able to show analytically that $0 \leqq \partial E(\widetilde{STK}(\tilde{\gamma},EBO)/\partial\lambda < \infty$ (see Lemmas B9 and B10), he was unable to derive any analogous result for the second partial derivative with respect to λ. Hence, there exists no good reason to dispute the validity of the exceptions above, i.e., evidence of concavity is not totally conclusive.

Figure 8. FIRST AND SECOND PARTIAL DERIVATIVES OF $E(\widetilde{STK}(\tilde{\gamma}, EBO))$ WITH RESPECT TO λ*

λ	MTBF	$\partial E_n(\widetilde{STK}(\tilde{\gamma}, EBO))/\partial\lambda$	$\partial^2 E_n(\widetilde{STK}(\tilde{\gamma}, EBO))/\partial\lambda^2$	$Pr(\gamma \leq a_n (EBO))$
.0005	2000	1397	-591,821	.82623
.0010	1000	1085	-564,857	.96980
.0015	667	836	-406,962	.99475
.0020	500	698	-145,775	.99909
.0025	400	671	7,806	.99984
.0030	333	683	25,860	.99997
.0035	286	688	-9,114	> .99999
.0040	250	675	-39,840	> .99999
.0045	222	652	-49,025	> .99999
.0050	200	629	-41,273	> .99999
.0055	182	612	-27,177	> .99999
.0060	167	601	-14,648	> .99999
.0065	154	596	-6,962	> .99999
.0070	142	594	-4,227	> .99999
.0075	133	591	-5,184	> .99999
.0080	125	588	-7,675	> .99999
.0085	118	584	-9,961	> .99999
.0090	111	578	-10,971	> .99999
.0095	105	573	-10,600	> .99999
.0100	100	568	-9,377	> .99999

*The second partial whose approximate values appear above is derived by Lemma B9. Also, all parameters and variables not tabulated above have the same values as for the example in Figure 7.

Figure 9. SUCCESSIVE DIFFERENCES OF $VAR(\widetilde{STK}(\tilde{\gamma}, EBO))$ AND $COV(\widetilde{STK}(\tilde{\gamma}, EBO), X_i(T))$, $i = 2, 3$, WITH RESPECT TO λ*

λ	MTBF	$\Delta Var_n(\widetilde{STK}(\cdot))/\Delta\lambda$	$\Delta Cov_n(\widetilde{STK}(\cdot), X_2(T))/\Delta\lambda$	$\Delta Cov_n(\widetilde{STK}(\cdot), X_3(T))/\Delta\lambda$	$Pr(\gamma \leq a_n(EBO))$
.0005	2000	755	1232	165	.82623
.0010	1000	175	671	103	.96980
.0015	667	-48	267	70	.99475
.0020	500	-20	192	93	.99909
.0025	400	111	455	105	.99984
.0030	333	175	651	94	.99997
.0035	286	156	637	80	> .99999
.0040	250	105	511	73	> .99999
.0045	222	67	392	75	> .99999
.0050	200	58	340	81	> .99999
.0055	182	70	353	87	> .99999
.0060	167	87	401	87	> .99999
.0065	154	99	450	84	> .99999
.0070	142	103	482	78	> .99999
.0075	133	99	488	72	> .99999
.0080	125	91	472	68	> .99999
.0085	118	82	446	67	> .99999
.0090	111	76	421	67	> .99999
.0095	105	73	406	69	> .99999
.0100	100	74	401	71	> .99999

*As in Figure 8, all parameters and variables not tabulated above have the same values as for the example in Figure 7.

EXPLORING THE VARIANCE RANGE IN THE GENERAL CASE

We can now define an NLP framework analogous to IV.12 for exploring the variance range and corresponding power function range for a given value of the underlying unknown mean of the estimator (III.12 repeated),

$$\left.\begin{aligned} MLSC(T) = {} & K_0 \widetilde{STK}(\tilde{\gamma}, EBO) + (K_1/T)X_3(T) + \sum_{k=1}^{X_1(T)} ((K_2/T)Y_{1k}^{(1)} + (K_2/T)Y_{2k}^{(1)}) \\ & + \sum_{k=1}^{X_2(T)} ((K_2/T)Y_{1k}^{(2)} + (K_2/T)Y_{3k}^{(2)} + (K_3K_4/T)Y_{4k}) \\ & + \sum_{k=1}^{X_3(T)} ((K_2/T)Y_{1k}^{(3)} + (K_2/T)Y_{3k}^{(3)} + (K_3K_5/T)Y_{5k}), \end{aligned}\right] \quad V.33$$

where $\widetilde{STK}(\tilde{\gamma}, EBO)$ is defined by V.17 and the observed variables by III.1a. This framework is

$$\left.\begin{aligned} & \text{maximize (minimize) } \sigma^2_{MMLSC} = f_1(p_1,p_2,p_3,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\lambda) \\ & \text{subject to} \\ & \text{a.} \quad MMLSC \equiv g_1(p_1,p_2,p_3,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\lambda) = K, \\ & \text{b.} \quad p_1 + p_2 + p_3 = 1 \\ & \text{c.1.} \ LB_1 \leq p_1 \leq UB_1, \\ & \quad 2. \ LB_2 \leq p_2 \leq UB_2, \\ & \quad 3. \ LB_3 \leq p_3 \leq UB_3, \\ & \quad 4. \ LB_4 \leq \mu_1 \leq UB_4, \\ & \quad 5. \ LB_5 \leq \mu_2 \leq UB_5, \\ & \quad 6. \ LB_6 \leq \mu_3 \leq UB_6, \\ & \quad 7. \ LB_7 \leq \mu_4 \leq UB_7, \\ & \quad 8. \ LB_8 \leq \mu_5 \leq UB_8, \\ & \quad LB_i \geq 0, \ i = 1, 2, \ldots, 8, \end{aligned}\right] \quad V.34$$

where $f_1(\cdot)$ and $g_1(\cdot)$ in V.34 are defined as follows:

$$\left.\begin{aligned}
MMLSC = E(MLSC) &= g_1(p_1,p_2,p_3,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\lambda) \\
&= K_0\ E(\widetilde{STK}(\widetilde{\gamma},EBO)) + E(MLSC') \\
&= K_0 \sum_{i=0}^{\infty} i \sum_{\substack{m_2,m_3, \\ a_{i-1}(EBO)<K_6m_2+K_7m_3\le a_i(EBO)}} \left(e^{-p_2\lambda T}\ \frac{(p_2\lambda T)^{m_2}}{m_2!}\right)\left(e^{-p_3\lambda T}\ \frac{(p_3\lambda T)^{m_3}}{m_3!}\right) \\
&\quad + \lambda\ (K_1p_3+K_2\mu_1+K_2p_1\mu_2+K_2p_2\mu_3+K_2p_3\mu_3+K_3K_4p_2\mu_4+K_3K_5p_3\mu_5)
\end{aligned}\right] \quad V.35$$

$$\left.\begin{aligned}
Var(MLSC) &= f_1(p_1,p_2,p_3,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\lambda) \\
&= Var(MLSC') + K_0^2\ Var(\widetilde{STK}(\widetilde{\gamma},EBO)) \\
&\quad + (2/T)(K_0K_2\mu_1+K_0K_2\mu_3+K_0K_3K_4\mu_4)\ Cov(\widetilde{STK}(\widetilde{\gamma},EBO),\ X_2(T)) \\
&\quad + (2/T)(K_0K_1+K_0K_2\mu_1+K_0K_2\mu_3+K_0K_3K_5\mu_5)\ Cov(\widetilde{STK}(\widetilde{\gamma},EBO),\ X_3(T))
\end{aligned}\right] \quad V.36a$$

$$\left.\sigma^2_{MMLSC} \equiv \underset{MMLSC}{Var(MLSC)} = \begin{array}{l}\text{the variance of MLSC (VAR(MLSC)) given} \\ \text{its underlying unknown mean is MMLSC}\end{array}\right] \quad V.36b$$

and where Var(MLSC') is given by IV.10a., Var($\widetilde{STK}(\tilde{\gamma}$,EBO)) by V.21, Cov($\widetilde{STK}(\tilde{\gamma}$, EBO),$X_2$(T)) by V.29, and Cov($\widetilde{STK}(\tilde{\gamma}$,EBO),$X_3$(T)) by V.30. Expression V.36a is derived in Lemma B11.

All elements of Framework V.34 are intended to be completely analogous in definition and interpretation to corresponding elements of Framework IV.12. Constraints V.34c form a hypercube in S_8, and V.34b is a hyperplane passing through the hypercube in S_3. The intersection of V.34b and c (called CS) is convex. IV.34a is a smooth surface in S_9 which also intersects CS. As MMLSC now includes an n'th partial sum approximation of E($\widetilde{STK}(\tilde{\gamma}$,EBO)), its functional characteristics are considerably more complicated than those of MMLSC'. Our knowledge of these characteristics is limited to what we can conclude from the description of E($\widetilde{STK}(\tilde{\gamma}$,EBO)) in the previous section: namely, since E($\widetilde{STK}(\tilde{\gamma}$,EBO)) appears to be an increasing concave function of λ, and MMLSC' is linear in λ then MMLSC is also an increasing concave function of λ. Even less

can be said about the functional properties of the objective function, σ^2_{MMLSC}. While Var(MLSC') is linear in λ (IV.10), this is certainly not the case for σ^2_{MMLSC}. It is clear from the periodically increasing and decreasing partial derivatives of the other three root elements of σ^2_{MMLSC}--Var($\widetilde{STK}(\tilde{\gamma}$,EBO)), Cov($\widetilde{STK}(\tilde{\gamma}$,EBO),$X_2$(T)), and Cov($\widetilde{STK}(\tilde{\gamma}$,EBO),$X_3$(T))--illustrated by Figure 9 that σ^2_{MMLSC}, while appearing still to increase with respect to λ, has no convexity or concavity properties even with respect to λ.

Since proof of the existence of global optima, and guaranteed methods for finding them are even more elusive in the case of V.34 than they are in the case of IV.12, the NLP algorithm, SUMT, was used to attempt to find both maxima and minima. The results were generally inconclusive primarily because convergence of moments involving $\widetilde{STK}(\tilde{\gamma}$, EBO) was too slow for SUMT to compute efficiently. When n'th partial sum error thresholds were set low to guarantee good approximations, convergence was too slow, resulting in exponent overflow errors. When error thresholds were increased, resulting in fewer partial sum terms, the overflow was avoided but SUMT would not converge. Presumably, this is due to gradients which were functions of partial sum approximations changing too erratically because they did not include enough summation terms to be sufficiently accurate.

The other major deterministic approach used to explore the properties of V.34 (and subsequently, to gain insight into the range of β(MMLSC) (III.16) as a function of representative LSCC data) was purely heuristic - namely, enumeration of extreme point values of the objective function and setting the approximate maximum and minimum solutions to V.34 to be the largest and smallest, respectively, of this set of extreme point values. More specifically, the bounds of Constraints V.34c were chosen both to be realistic <u>and</u> to ensure that the extreme points of the intersection, CS, of V.34b and V.34c could be enumerated without difficulty. Next, the eight coordinates of each

extreme point were substituted into V.34a and this equality constraint solved for λ. Since MMLSC was shown to be an increasing function of λ and is clearly continuous, either a direct (e.g., binary) search can be used to find this λ value or, in some instances, a Taylor series approximation (substituting the first two (or three) terms of the series and solving the resulting linear (quadratic) equation directly). Finally, the nine resulting coordinates were substituted into the objective function.

This extreme point heuristic approach appears to be a fairly adequate approximating tool based on results described in Chapter IV. In the simpler MLSC' case, there is considerable empirical evidence to suggest that the global maximum variance is at one of these extreme points and some similar evidence regarding the global minimum. Presumably, although Var(MLSC) and MMLSC are much more complex functions than Var(MLSC') and MMLSC', respectively, they are still similar enough in shape and continuity features to make a good argument for the extreme point approach in the general case. Results to be described in Chapter VI provide further justification for this approach.

The numerical results of this approach lead to the same conclusions as those illustrated by examples given in reference 4. For reasonable test lengths, T, the β(MMLSC) range at any given value of MMLSC appears to be acceptably small in width. However, values of these β(MMLSC)'s, e.g., as represented by the mean of the range, are quite large; e.g., Pr(Type i error) $> .3$, $i = 1, 2$.[5]

SUMMARY AND EVALUATION OF ANALYSIS RESULTS

The statistical properties of the estimators, MLSC' and MLSC, described in this and the previous two chapters are clearly complex. The mean and

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[5] Further discussion of the implications of these numerical results is found in reference 4 as well as Chapters VI and VII.

variance of the simpler estimator, MLSC', can be derived and computed precisely. However, when a reasonable set of bounds such as IV.12c is defined with respect to the nine individual unknown parameters, p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, μ_5, and λ, the estimates of which are arguments of MLSC', least upper and greatest lower bounds on its nonunique variance (and power function) cannot be determined with certainty. While there is considerable empirical evidence that extreme point enumeration may yield good approximations of global optima to Problem IV.12 and is a computationally feasible solution approach, enumeration algorithms that can be proven to converge to global optima have not been found. Nonetheless, further exploration of such properties as the convexity features of an exponential transformation of IV.12 may yet lead to computationally feasible methods for finding these variance bounds with certainty.

The mean and variance of the more complex estimator, MLSC, used in recent LSCC applications, can be formulated precisely but must be computed approximately due to the dependence of these functions on an infinite series. Furthermore, least upper and greatest lower bounds on the MLSC's nonunique variance (and power function) are even more difficult to determine with certainty than in the MLSC' case. While useful enumeration based heuristics described herein for approximating global optima to V.34 have been developed for specific limited examples, the difficult tasks of (1) making these heuristics computationally feasible in the general case and (2) bounding their errors remain.

While the NLP frameworks (IV.12 and V.34) to which the above computational results apply have considerable intuitive appeal and meaning in an LSCC applications environment (see discussion following IV.12), they contain some weaknesses. Consider the case of V.34 for purposes of illustration. Experimental results to date suggest that as K (the RHS of V.34a), the value

assumed by the underlying unknown mean, MMLSC, is varied over its domain, the resulting maximum and minimum power functions values, β_{max}(MMLSC) and β_{min}(MMLSC), form a continuously bounded power function envelop of the form shown in Figure 10. Moreover, computational results suggest that for a given value of K, $\beta_{max}(\cdot)$ and $\beta_{min}(\cdot)$ are attained at minimum and maximum values of λ, respectively (recall that λ is implicitly bounded in V.34). This seems reasonable in view of the fact that Var(MLSC) is linear in λ but a higher order function of other of its arguments. Furthermore, these results also indicate that as K increases, increases in $\beta_{max}(\cdot)$ and $\beta_{min}(\cdot)$ are caused exclusively by decreases in *minimum* and *maximum* values of λ, respectively, as opposed to changes in optimal values of the remaining eight unknown underlying parameter values (i.e., movement away from a given extreme point of CS).[6] This phenomenon highlights the fact that V.34, as currently formulated, requires the assumption that the difference between any two values that the underlying unknown MMLSC might assume, particularly when the two values are not very close, is caused primarily by changes in the value of the underlying unknown λ. In other words, V.34 does not adequately allow for major differences in the value that MMLSC can assume to be explained in terms of changes in the other eight parameter values. Further, the existence of the continuously bounded power function envelop in Figure 10 requires this assumption (analogous results apply to IV.12)

It is for these reasons that earlier descriptions of research results related to IV.12 and V.34 did not address the impact of variations in K but rather focused on the bounding of risk at one, two, or perhaps more discrete

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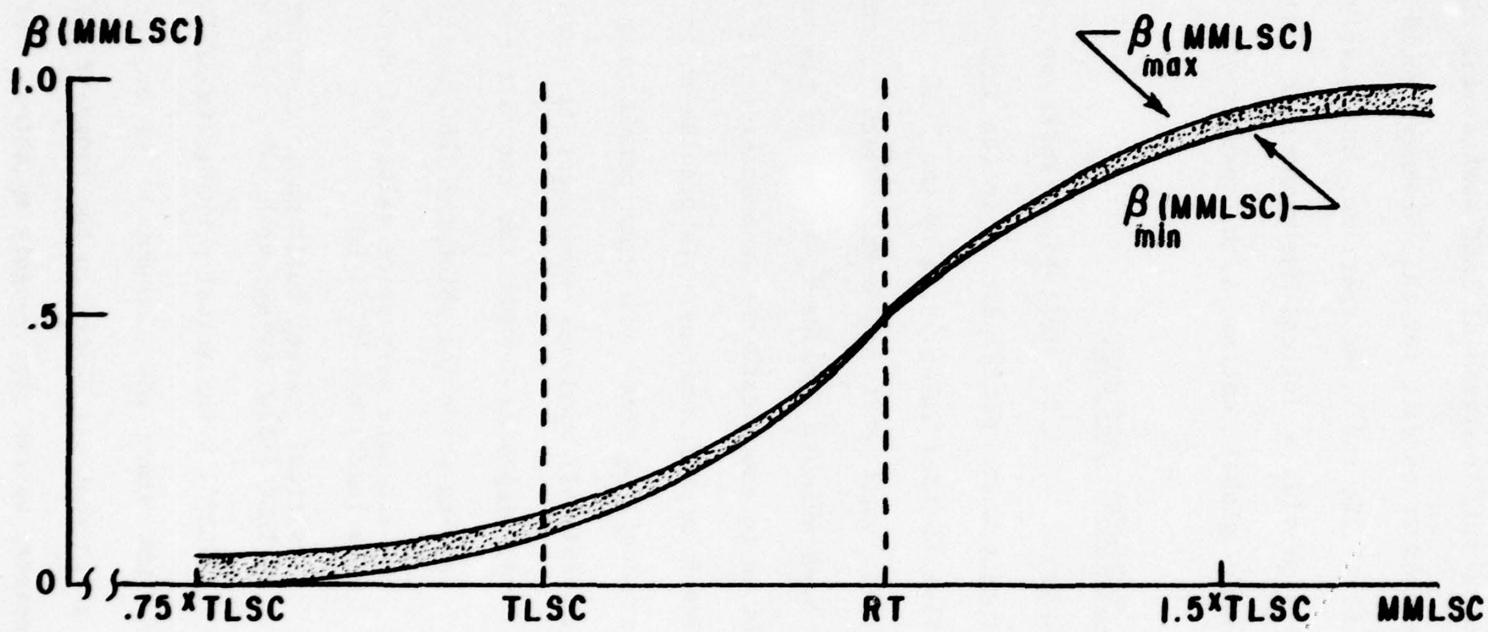
[6]The hypothesis that the global maximum and minimum to V.34 both are attained at an extreme point of CS (in S_8) and remain fixed at this extreme point for different values of K was strongly substantiated by the author's experimental results but remains to be proved or disproved.

Figure 10. **THE POWER FUNCTION ENVELOPE CORRESPONDING TO PROBLEM V.34**



values of K. In this regard, IV.12 and V.34 have limited intuitive appeal as frameworks for a full analysis of LSCC statistical risks. One way to enhance the two frameworks to correct for this weakness would be to make the upper and lower bounds of IV.12c and V.34c specific, intuitively acceptable functions of K. This would provide a logical framework for analysis of the behavior of IV.12 and V.34 global optima with respect to K (recall Chapter IV, Footnote 5).

TREATMENT OF EQUIPMENT UNIT COST

Our discussion of LSCC statistical risks so far has assumed that the ultimate equipment unit cost appearing in the MLSC value does not differ from its target value UC used in building up the TLSC. In most recent LSCC applications, however, unit cost has been subject to change by the Contractor, i.e., he has been able to include it as one of his trade-off parameters. The effect of this is to complicate the assessment of LSCC risks.

One approach to ameliorating this problem is to redefine the target unit price, UC, as a ceiling price and hence permit only reductions in unit price in contractor trade-off analyses subsequent to negotiation of the TLSC. As shown below, this approach bounds the contractor's statistical risk. Let

$$\left.\begin{array}{lll} UC & = & \text{ceiling price (of which the TLSC is a function),} \\ UC^* & = & \text{the ultimate unit price value (of which the MLSC is a function), } UC^* \leq UC, \\ TLSC^* & = & \text{a "virtual" target built up in a manner identical to TLSC (II.4) except with } UC^* \text{ replacing UC,} \\ \alpha^* & = & \beta(TLSC^*) = \text{the actual probability of Type I error.} \end{array}\right] \quad \text{V.37}$$

Then TLSC* ≦ TLSC since LSC (Figure 1) is an increasing function of UC. Furthermore, given that all values in the range of β(MMLSC) generally increase as MMLSC increases, we can say, roughly speaking, that

$$\alpha^* \leq \alpha, \qquad \text{V.38}$$

i.e., the LSCC value of contractor statistical risk is an upper bound on ultimate contractor risk.

This interpretation of unit cost has appeal because it is entirely reasonable to expect UC* < UC. Namely, the contractor should have little inclination to make UC* > UC within the framework of the LSCC. However, if values of MTBF, NRTS, etc., in equipment to be tested are likely to produce a high MLSC, he <u>may</u> be inclined to offset this phenomenon by reducing the unit price, UC*, of his first lot of spare items with respect to which UC* is defined, in order to produce an MLSC value less than the remedy threshold, RT.

The weakness of this approach is that the Government's capability to verify the equipment's LSC behavior is diminished. Since TLSC* < TLSC, RT > TF * TLSC* and TLSC', the Government's rejection target, tends to be further to the right of the actual Contractor target, TLSC*, than desired. This and other approaches to address UC behavior are worthy of further research.

VI. A MONTE CARLO SIMULATION MODEL FOR LSCC STATISTICAL RISK ESTIMATION

While the analytical results in the preceding chapters provide significant insights into the risk characteristics of the LSCC, they do not provide any computational procedure for calculating or approximating the range of β(MMLSC) in the multiple item or aggregate case (MLSC analog to IV.15 for MLSC'). Since there are several present applications of this case and it has properties suggesting reduced risk, the author chose to develop a Monte Carlo simulation model to approximate these risks. While this model lacks the precision of the analysis previously described, it does provide both useful numerical results for the aggregate case and some important additional insights into an alternative way to formulate the LSCC risk framework.[1]

This model is briefly described in two parts below. The first part describes the model structure for computing risks for a single equipment item and the second part describes the structural enhancements needed to deal with the multiple item case.

THE SINGLE ITEM MODEL

The objective of this model is to generate a sample prior distribution of σ^2_{MMLSC} and hence β(MMLSC) corresponding to a given value K of the MMLSC as a function of prior distributions of the nine unknown underlying parameters, p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, μ_5, and λ. The basis of the model is the constraint set of V.34, shown again below for convenience.

$$\left.\begin{array}{ll} \text{a.} & \text{MMLSC} \equiv g_1(p_1, p_2, p_3, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \lambda) = K \\ \text{b.} & p_1 + p_2 + p_3 = 1 \end{array}\right|$$

[1]This simulation model provided the numerical results discussed in reference 4.

$$
\left.
\begin{array}{l}
\text{c.1. } LB_1 \leqq p_1 \leqq UB_1, \\
\text{2. } LB_2 \leqq p_2 \leqq UB_2, \\
\text{3. } LB_3 \leqq p_3 \leqq UB_3, \\
\text{4. } LB_4 \leqq \mu_1 \leqq UB_4, \\
\text{5. } LB_5 \leqq \mu_2 \leqq UB_5, \\
\text{6. } LB_6 \leqq \mu_3 \leqq UB_6, \\
\text{7. } LB_7 \leqq \mu_4 \leqq UB_7, \\
\text{8. } LB_8 \leqq \mu_5 \leqq UB_8, \\
\qquad LB_i \geqq 0,\ i = 1, 2, \ldots, 8,
\end{array}
\right] \qquad \text{VI.1}
$$

where MMLSC is again defined by V.35.

Under this approach, prior distributions for p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, and μ_5 are defined on the closed intervals VI.1.c.1. through c.8., respectively. The process for generating a single variate of the sample prior distribution of σ^2_{MMLSC} has the following steps:

1. A set of eight variates, p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, and μ_5, is generated under appropriate prior distributions subject to constraint VI.1b. being satisfied.

2. These eight prior distribution variates are substituted into VI.1a, which is subsequently solved for the λ variate.

3. A variate of the sample prior distribution of σ^2_{MMLSC} (and subsequently of $\beta(MMLSC)$) is computed from the nine underlying parameter variates using V.36.

A sample prior distribution of σ^2_{MMLSC} is built up by continuing these iterations until sample distribution statistics stabilize.

The two parameter beta distribution was chosen as the prior distribution for p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, and μ_5 because it is easily defined and scaled to any given finite closed interval and subjective opinions regarding its mean

and variance can be easily characterized by appropriate values of its two parameters.[2]

Random p_1, p_2, and p_3 variates such that VI.1b was satisfied were generated using a heuristic approach sometimes referred to as the "hit-or-miss" process.[3] Namely, two of the three variables were generated, say (p_1, p_2) for purposes of illustration. If $p_1 + p_2 < 1$, the p_3 variate was defined to be $1 - p_1 - p_2$. If $p_1 + p_2 \geq 1$, both variates were discarded and a new set generated, and so on. Clearly, by VI.1b, the prior distribution of p_3 is a function of the priors of p_1 and p_2. But since this function is not easily derived, the p_3 variate was generated indirectly using the "hit-or-miss" process. The user can increase his control of the constrained variate (p_3 in the case above) prior distribution through sensitivity analysis of the <u>impact of changes</u> in the p_1 and p_2 priors on the p_3 sample prior, and making permanent changes in the p_1 and p_2 priors, accordingly.

The prior distributions of p_1, p_2, and p_3 are obviously not independent. Thus, ideally one would like to know $f_{P_1}(p_1)$, the prior on p_1 and $f_{P_2|P_1}(p_2|p_1)$, the conditional prior on P_2 given $P_1 = p_1$. Also, $p_3 = 1 - p_1 - p_2$. Clearly, $f_{P_2|P_1}(p_2|p_1)$ would reflect the fact that $p_2 \varepsilon (0, 1-p_1)$ rather than assuming $p_2 \varepsilon (0,1)$. This could be established by either establishing some relationship between p_1 and p_2 or by simply "scaling" the prior to the new range.

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[2]The beta distribution is further described in reference 17, pp. 89, 90 and reference 20, pp. 216-221.

[3]See reference 10, pp. 53-55, 65, 128. This approach is commonly used when random variates must be generated subject to constraints.

Choosing the λ variate to be the λ value which solves VI.1a determines the sample prior distribution for λ in the same manner as the p_3 sample prior was determined. As in the case of the p_3 prior the λ prior as a function of the other eight priors is not easily derived and so its sample prior can be controlled and modified again only indirectly via sensitivity analysis.

The above model approach can be described in geometric terms as generating a large number of points on the surface of the function σ^2_{MMLSC} for values of its nine arguments within Constraint Region VI.1. One interesting result obtained with this model was that all σ^2_{MMLSC} values generated were within the greatest lower and least upper bounds determined by the extreme point enumeration approach described in Chapter V for all example problems used. This provides further support for the hypothesis that both the global maximum and minimum to V.34 are attained at an extreme point of the convex constraint set CS in S_8.

Experimentation with this model indicated that in order for model results to be most meaningful, two conditions were necessary:

a) Along with K, the RHS of VI.1a, a set of specific "target" values of the nine parameters which yielded the given value, K, when substituted in V.35 (MMLSC) had to be defined (for <u>all</u> values of K of potential interest).[4]

b) The two parameters of each of the prior distributions directly controlled by the user had to be chosen so that each prior distribution mean was approximately equal to the target value of the parameter.

VI.2

Given that parameter targets are explicitly defined as specified by VI.2.a and prior distribution parameter values are chosen such that one or more prior means are <u>not</u> at their targets, then the sample mean of the prior distribution of λ may be far from the target value of λ and the resulting sample prior

[4]These are typically defined already for K = TLSC. Reference 4 also defines a set formally for K = TLSC', the Government's "rejection target."

distribution of σ^2_{MMLSC} centered around an unrealistically high or low value. If, in addition, targets are not defined in accordance with VI.2.a, then there is no longer any standard by which to evaluate the prior distribution of λ, e.g., no target with which to compare its sample mean. It follows that different choices of parameter values for the prior distributions of p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, and μ_5 can result in widely different sample prior distributions of σ^2_{MMLSC}, each without much meaning.

The model framework also has the property that, as long as Conditions VI.2 above are fulfilled, the prior distribution of σ^2_{MMLSC} is not particularly sensitive to differences in the variances of the prior distributions chosen with respect to VI.1. This may result from Constraint VI.1a having a canceling effect in increases in prior variances of parameters bounded by VI.1c. This property eases the subjective choice of prior distribution parameter values.

A final important property of this model is that, as long as Conditions VI.2 above are fulfilled, the upper and lower bounds UB_i and LB_i, $i = 1, 2, \ldots, 8$, can be changed quite considerably without any significant impact on the sample prior distribution of σ^2_{MMLSC}. In other words, the choice of the size and shape of the eight-dimensional hypercube in VI.1 is very much peripheral to the choice of prior distribution parameters. This model property contrasts sharply with the central role of the constraint region boundary in the methods examined for solving IV.12 and V.34. The approach taken in defining and trying to solve IV.12 and V.34 was simply to bound underlying unknown parameters and compute the resulting range of risks. This approach is precise and the only subjective information that it calls for is these bounds. In contrast, the above model is both imprecise and requires much more subjectively determined information in the form of prior distributions of underlying unknown parameters. However, as long as Conditions VI.2 above are

fulfilled, the increased information content in these priors makes the length of the interval over which they are defined almost a moot point.

THE AGGREGATE MODEL

Recall that the approach taken in the multiple item case is to compute a target LSC for each item in terms of a CMF such as that in Figure 1 and to sum these targets to an aggregate target which becomes the formal basis for the LSCC. To describe the aggregate model structure, we define the following notation, analogous to that in the last section of Chapter IV:

a. N = the total number of equipment items for which the target LSC is defined,

b. $MLSC_i$ = the measured LSC for equipment item i, $i = 1, 2, \ldots, N$,

c. $MLSC$ = the total measured LSC (the aggregate estimator) $= \sum_{i=1}^{N} MLSC_i$,

d. $MMLSC_i \equiv E(MLSC_i),\ i = 1, 2, \ldots, N$,

e. $MMLSC = E(MLSC) = \sum_{i=1}^{N} MMLSC_i$, VI.3

f. $TLSC_i$ = the target LSC for equipment item i, $i = 1, 2, \ldots, N$,

g. $TLSC$ = the aggregate target LSC $= \sum_{i=1}^{N} TLSC_i$,

h. σ^2_{MMLSC} = (the variance of the MLSC given its underlying unknown mean is MMLSC) $= \mathrm{Var}_{MMLSC}(MLSC)$

$$= \sum_{i=1}^{N} \mathrm{Var}_{MMLSC_i}(MLSC_i) \,\Big|\, \sum_{i=1}^{N} MMLSC_i = MMLSC .$$

The basis for the multiple item simulation model is a constraint framework in many ways analogous to IV.15:

$$\left.\begin{array}{lll}
\text{a.} & MMLSC = \sum_{i=1}^{N} MMLSC_i = K & \\
\text{b.} & p_{i1} + p_{i2} + p_{i3} = 1, & i = 1, 2, \ldots, N, \\
\text{c.1.} & LB_{i1} \leqq p_{i1} \leqq UB_{i1}, & i = 1, 2, \ldots, N, \\
\text{2.} & LB_{i2} \leqq p_{i2} \leqq UB_{i2}, & i = 1, 2, \ldots, N, \\
\text{3.} & LB_{i3} \leqq p_{i3} \leqq UB_{i3}, & i = 1, 2, \ldots, N, \\
\text{4.} & LB_{i4} \leqq \mu_{i1} \leqq UB_{i4}, & i = 1, 2, \ldots, N, \\
\text{5.} & LB_{i5} \leqq \mu_{i2} \leqq UB_{i5}, & i = 1, 2, \ldots, N, \\
\text{6.} & LB_{i6} \leqq \mu_{i3} \leqq UB_{i6}, & i = 1, 2, \ldots, N, \\
\text{7.} & LB_{i7} \leqq \mu_{i4} \leqq UB_{i7}, & i = 1, 2, \ldots, N, \\
\text{8.} & LB_{i8} \leqq \mu_{i5} \leqq UB_{i8}, & i = 1, 2, \ldots, N, \\
\text{9.} & LB_{i9} \leqq \lambda_i \leqq UB_{i9}, & i = 1, 2, \ldots, N-1, \\
 & (LB_{ij} \geqq 0,\ i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, 9). &
\end{array}\right] \qquad \text{VI.4}$$

The computational approach in the aggregate model is largely analogous to that of the single item model. Again the objective is to generate a sample prior distribution of σ^2_{MMLSC} and hence $\beta(MMLSC)$ corresponding to a given value, K, of the MMLSC. Under this approach, prior distributions are defined for p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, μ_5, <u>and λ</u> on the appropriate intervals defined by VI.4c for each of the first (N-1) equipments and for only the first eight parameters for the N'th equipment. The process for generating a simple variate of the sample prior distribution of σ^2_{MMLSC} has the following steps:

1. A set of variates, (p_{i1}, p_{i2}, p_{i3}, μ_{i1}, μ_{i2}, μ_{i3}, μ_{i4}, μ_{i5}, λ_i), i = 1, 2, ..., N-1, is generated, again using the "hit or miss" heuristic to ensure that VI.4b is satisfied for each item.

2. Values of the variate, $MMLSC_i$, i = 1, 2, ..., N-1, are computed from the variate sets in (1) and the quantity, $\sum_{i=1}^{N-1} MMLSC_i$, determined.

3. Assuming $\sum_{i=1}^{N-1} MMLSC_i < K$, a set of variates, ($p_{N1}$, p_{N2}, p_{N3}, μ_{N1}, μ_{N2}, μ_{N3}, μ_{N4}, μ_{N5}) is generated for the N'th equipment (again such that VI.4b is fulfilled) and the value of λ_N such that $MMLSC_N = K - \sum_{i=1}^{N-1} MMLSC_i$ is determined (if $\sum_{i=1}^{j} MMLSC_i \geqq K$ for one of more values of j, j = 1, 2, ..., N-1, then all parameter variates generated in this iteration are discarded and the iteration begun anew (a second application of the "hit-or-miss" heuristic)).

4. Values of the variate, $Var_{MMLSC_i}(MLSC_i)$, i = 1, 2, ..., N are computed from the variate sets generated in (1) and (3) above, and summed according to VI.3h. to obtain a σ^2_{MMLSC} variate and $\beta(MMLSC)$ variate.

Iterations are continued until statistics of the sample prior distributions of σ^2_{MMLSC} and $\beta(MMLSC)$ stabilize.

Again the beta distribution is used as the prior for p_{i1}, p_{i2}, p_{i3}, μ_{i1}, μ_{i2}, μ_{i3}, μ_{i4}, and μ_{i5}, i = 1, 2, ..., N, and subjective estimates of its two parameters which completely specify its mean and variance must be provided with respect to each of the eight unknown parameters for each equipment item. The prior distribution of λ_i, i = 1, 2, ..., N-1, used for the aggregate model is the sample prior distribution of λ_i generated by the single item model when applied to each of the N-1 items individually. This approach guarantees that for N-1 of the N items, the prior distributions of p_1, p_2, p_3, μ_1, μ_2, μ_3, μ_4, μ_5, <u>and λ</u> will be approximately the same in the aggregate model as in the single item model while permitting the prior $MMLSC_i$ value, i = 1, 2, ..., N-1, to be constant as defined in the single item model and to vary as defined in

the multiple item model. Analogous to the case of λ in the single item model, the prior distribution of λ_N in the aggregate model is completely determined by the remaining 9(N-1) + 8 prior distributions and the manner in which Constraint VI.4a links them together.

Analogous to the single item model, the aggregate model, in order for its results to be meaningful, requires that (1) the RHS of VI.4a, K, be explicitly defined in terms of "target" values of the parameters p_{i1}, p_{i2}, p_{i3}, μ_{i1}, μ_{i2}, μ_{i3}, μ_{i4}, μ_{i5}, λ_i, $i = 1, 2, \ldots, N$ and that the means of all prior distributions be approximately equal to their corresponding targets (see Conditions VI.2 in the previous section).

As illustrated by the aggregate example in reference 4, the aggregate model computational results demonstrate that aggregation of cost targets in the manner described above and in the last section of Chapter IV can lead to significantly reduced statistical risk due to independence among the $MLSC_i$ estimators, $i = 1, 2, \ldots, N$.

VII. FURTHER LSCC RESEARCH

THE LSCC AS A COST REDUCTION TOOL

One of the foremost difficulties in motivating defense system and equipment contractors to develop and produce minimum LCC hardware is that it is simply not practical to extend a development and production contract life span well into the hardware deployment phase (e.g., 5-10 years). Hence, the Contractor cannot easily be held accountable in his contract for both the production cost and operating and support (O&S) cost impact of his design.

The LSCC is one approach to solving this problem. It incorporates a CMF as a surrogate for actual incremental LSCs ultimately incurred over an item's full life. The CMF serves as the basis for defining a contractual LSC target (TLSC) and for development of an estimate (MLSC) of ultimate LSCs as summarized by the CMF. A key element of the LSCC concept is that the estimate is developed from LSC performance data collected over a relatively short period, e.g., 6 months to 2 years, which can reasonably be incorporated within the contract life span. A comparison of target and estimated surrogate LSCs can result in additional contract awards or requirements for contractor remedies.

While the LSCC is an innovative contracting tool, there are some very basic questions, aside from the issue of statistical uncertainty, associated with its application, which remain unresolved:

a) Since the CMF does not represent actual incremental costs, can it be structured to simulate, duplicate, or approximate adequately the functional dependencies of actual incremental LSC elements upon contractor-controlled design parameters?[1]

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[1] Reference 15 addresses certain aspects of this question in a very comprehensive discussion of the accuracy of several LCC models including the Air Force LSC Model.

b) Given subjective evidence that sufficient approximation of these functional dependencies is achievable, can more objective or perhaps quantitative measures of approximation adequacy be developed? VII.1

c) Assuming (a) and (b) above can be answered affirmatively, does the LSCC in its totality as a contracting approach succeed in motivating the hardware contractor to allocate his design efforts and funds among design parameters in proportion to their impact on LSCs in the manner suggested by the CMF?

Presently, there is some doubt in the military cost analysis community whether all three of these questions can ever be answered affirmatively. While they have been examined, debated, and researched to some degree, they are of sufficient importance to warrant further research.

STATISTICAL RISK RESEARCH TO DATE: FUTURE POTENTIAL

The research into the statistical risk properties of the LSCC documented herein has generated important insights into these properties with relatively little effort (< two man years). Chapters III-V illustrate numerous heretofore not well understood LSCC properties of a very basic nature. Chapter VI describes a comprehensive approximate model which can be used to measure statistical risks associated with various existing LSCCs with considerable accuracy.

Hence, a great deal has been learned with a minimum of effort. Furthermore, several areas of potentially fruitful research have been isolated. For example, statistical risk behavior as a function of changes in the CMF due to (1) inclusion of alternative sets of LSC elements or (2) tailoring of the CMF to different equipment types needs to be investigated. Methods of modifying the MLSC estimator definition such that statistical risks are reduced at the expense of the ability to verify various sets of design parameter interactions (e.g., MTBF-NRTS) are worthy of further attention. Additional avenues for further research can be readily defined.

In view of this progress to date and ease with which potentially fruitful research areas can be defined, it is certainly not appropriate at this time to conclude that the LSCC's statistical properties diminish its potential usefulness. Indeed, it is quite possible that LSCCs incorporating appropriately modified CMFs, test plans, and terms and conditions can be formulated and applied effectively in the future.

A SCENARIO FOR FURTHER RESEARCH

Further research into the LSCC should be oriented along two lines:

a) Pursuit of answers to Questions *VII.1* to learn more about (1) the adequacy of CMF approximations of actual incremental LSC dependencies on design parameters, and (2) the contractual elements necessary to provide for proper contractor design motivation with respect to LSCs.

b) Further analysis of statistical risk properties along the lines illustrated in the previous section to determine if more practicable verification test plans for the LSCC can be found.

Furthermore, these two avenues of research should be pursued concurrently because they are equally important and very much intertwined. Indeed, as the CMF is modified, changes occur in both (1) the nature of the approximation of actual incremental cost functional dependencies upon design parameters and (2) underlying statistical risk characteristics. Both modes of change work individually and interactively to affect the characteristics of the ultimate cost reduction incentive to be transmitted.

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APPENDIX A

DERIVATION OF MEAN AND VARIANCE PROPERTIES OF MLSC´

<u>Lemma A1</u>:

a) $X_i(T)$ is Poisson distributed with parameter $p_i\lambda T$, $i = 1, 2, 3$.

b) $X_1(T)$, $X_2(T)$, and $X_3(T)$ are mutually independent.

<u>Proof</u>:

a) We perform the proof for $X_1(T)$ only and extend it by symmetry to $X_2(T)$ and $X_3(T)$.

Pr($X_1(T)$ failures in time T requiring repair in place)

$$= \sum_{i=X_1(T)}^{\infty} \Pr \left\{ \begin{matrix} X_1(T) \text{ failures in time T requiring repair} \\ \text{in place} \mid i \text{ failures in time T} \end{matrix} \right\} \cdot \Pr \left\{ \begin{matrix} i \text{ failures} \\ \text{in time T} \end{matrix} \right\}$$

$$= \sum_{i=X_1(T)}^{\infty} \left[\cancel{i!}/(X_1(T)!(i-X_1(T))!)\right] p_1^{X_1(T)} (1-p_1)^{(i-X_1(T))} e^{-\lambda T} (\lambda T)^i/\cancel{i!}$$

$$= e^{-\theta} \sum_{i=n}^{\infty} \left[1/(n!(i-n)!)\right] p_1^n (1-p_1)^{i-n} \theta^i$$ (where we have set $\theta = \lambda T$ and $n = X_1(T)$ for simplicity of exposition)

$$= e^{-\theta} \sum_{m=0}^{\infty} (1/(n!m!)) p_1^n (1-p_1)^m \theta^{m+n}$$ (where we have set $m = i - n$)

$$= e^{-\theta} p_1^n (\theta^n/n!) \sum_{m=0}^{\infty} (1-p_1)^m \theta^m/m!$$

$$= e^{-\theta}((p_1\theta)^n/n!)\, e^{\theta-\theta p_1}$$ (since $\sum_{k=0}^{\infty} X^k/k! = e^X$)

$$= ((p_1\theta)^n/n!)\, e^{-\theta p_1}$$

$$= (p_1\lambda T)^{X_1(T)}\, e^{-p_1\lambda T}/X_1(T)!$$ Q.E.D.

b) (also proven in reference 10, pp. 216,217)

$\Pr(X_1(T)=n_1,\ X_2(T)=n_2,\ X_3(T)=n_3)$

$= \Pr(X_1(T)=n_1,\ X_2(T)=n_2,\ X_3(T)=n_3 \text{ and } \underbrace{X_1(T)+X_2(T)+X_3(T)}_{\equiv X(T)} = \underbrace{n_1+n_2+n_3}_{\equiv n})$

$= \Pr(X_1=n_1,\ X_2=n_2,\ X_3=n_3 | X=n)\ \Pr(X=n)$

$= (\cancel{X!}/(X_1!X_2!X_3!))p_1^{X_1}\ p_2^{X_2}\ p_3^{X_3}\ (\lambda T)^X\ e^{-\lambda T}/\cancel{X!}$

by the multinomial law (see reference 11, p. 108).

$$= \left((p_1\lambda T)^{X_1}\ e^{-p_1\lambda T}\Big/X_1!\right)\left((p_2\lambda T)^{X_2}\ e^{-p_2\lambda T}\Big/X_2!\right)\left((p_3\lambda T)^{X_3}\ e^{-p_3\lambda T}\Big/X_3!\right)$$

Q.E.D.

<u>Lemma A2</u>: $E\left(\sum_{k=1}^{X_i(T)} Y_{jk}\right) = \mu_j p_i \lambda T,\ i = 1, 2, 3;\ j = 1, 2, .\ ., 5.$

<u>Proof</u>: For simplicity of exposition, let $\sum_{k=1}^{X_i(T)} Y_{jk}$ be replaced by $S_N \equiv \sum_{n=0}^{N} V_n$, where we have defined the following quantities:

1) $\{V_n,\ n = 1, 2, . . .\}$ is a sequence of independent and identically distributed (i.i.d.) random variables assuming values on $[0, \infty)$ with $E(V_n) = \mu$ and $Var(V_n) = \sigma^2$.

2) N is a random variable assuming values on $[0, 1, . . .)$ $(S_o \equiv 0)$ with $E(N) = p\lambda T$ and N is independent of $\{V_n,\ n = 1, 2, . . .\}$.

Then, $E(S_N) = \sum_{n=0}^{\infty} \Pr(N=n)\ E(S_N | N = n)$

$= \sum_{n=0}^{\infty} \Pr(N = n)\ E(V_1 + V_2 + . . . + V_N | N = n)$

$= \sum_{n=0}^{\infty} \Pr(N = n)\ n\mu$

$$= \mu \sum_{n=0}^{\infty} n \Pr(N = n)$$

$$= \mu p \lambda T \qquad \text{Q.E.D.}$$

Other proofs can be found in reference 9, p. 56 and reference 14, p. 40.

<u>Lemma A3</u>: $\text{Var}\left(\sum_{k=1}^{X_i(T)} Y_{jk}\right) = p_i \lambda T(\mu_j^2 + \sigma_j^2)$, $i = 1, 2, 3;\ j = 1, 2, 3, . . , 5.$

<u>Proof</u>: We use the notation developed in the proof of Lemma A2. To show that $\text{Var } S_N = p\lambda T(\mu^2 + \sigma^2)$, we use the important formula (proven in reference 9, p. 55) that, given two random variables, Y and X, then

$$\text{Var } Y = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X)),$$

i.e., the variance is equal to the mean of the conditional variance plus the variance of the conditional mean. By this, we have

$$\begin{aligned}
\text{Var } S_N &= E(\text{Var}(S_N|N)) + \text{Var}(E(S_N|N)) \\
&= E(\text{Var}(V_1 + V_2 + \ldots + V_N|N)) + \text{Var}(E(V_1 + V_2 + \ldots + V_N|N)) \\
&= E(N \text{ Var } V_n) + \text{Var } (N\ E(V_n)) \\
&= E(N) \text{ Var } V_n + E^2(V_n) \text{ Var } N \\
&= p\lambda T\ \sigma^2 + p\lambda T\ \mu^2 \\
&= p\lambda T\ (\sigma^2 + \mu^2)
\end{aligned}$$

Q.E.D.

<u>Lemma A4</u>: $\text{Cov}\left(X_i(T), \sum_{k=1}^{X_i(T)} Y_{jk}\right) = \mu_j\ p_i \lambda T$, $i = 1, 2, 3;\ j = 1, 2, . . , 5.$

<u>Proof</u>: Using the notation of Lemma A2, we want to show that $\text{Cov}(N, S_N) = \mu\ p\lambda T$.

$$\begin{aligned}
\text{Cov}(N, S_N) &= E(NS_N) - E(N)E(S_N) \\
&= E(NS_N) - E^2(N)E(V_n) \quad \text{by Lemma A2}
\end{aligned}$$

But $E(NS_N) = \sum_{n=1}^{\infty} Pr(N=n)\ E(NS_N | N=n)$

$= \sum_{n=1}^{\infty} Pr(N=n)\ E(n(V_1+V_2+.+V_n) | N=n)$

$= \sum_{n=1}^{\infty} Pr(N=n)\ n^2\ E(V_n)$

$= E(V_n) \sum_{n=1}^{\infty} n^2\ Pr(N=n)$

$= E(V_n)\ E(N^2)$

Thus, $Cov(N, S_N) = E(V_n)\ E(N^2) - E^2(N)\ E(V_n)$

$= E(V_n)\ (E(N^2) - E^2(N))$

$= E(V_n)\ \text{Var}\ N$

$= \mu p\lambda T$ Q.E.D.

<u>Lemma A5</u>: $Cov\left(\sum_{k=1}^{X_i(T)} Y_{mk},\ \sum_{k=1}^{X_i(T)} Y_{nk}\right) = \mu_m\mu_n p_i\lambda T,$
$i=1,2,3;\ m=1,2,..,5;\ m\neq n.$

<u>Proof</u>: Define two sequences of i.i.d. variables, $\{V_{1n}, n=1, 2, \ldots\}$ with $E(V_{1n}) \equiv \mu_1$ and $\{V_{2n}, n=1, 2, \ldots\}$ with $E(V_{2n}) \equiv \mu_2$. Using this notation, we will show that $Cov\left(\sum_{n=1}^{N} V_{1n},\ \sum_{n=1}^{N} V_{2n}\right) = \mu_1\mu_2 p\lambda T.$

By Lemma A2, we know that

$$Cov\left(\sum_{n=1}^{N} V_{1n},\ \sum_{n=1}^{N} V_{2n}\right) = E\left((\sum_{n=1}^{N} V_{1n})(\sum_{n=1}^{N} V_{2n})\right) - E\left(\sum_{n=1}^{N} V_{1n}\right) E\left(\sum_{n=1}^{N} V_{2n}\right)$$

$$= E\left((\sum_{n=1}^{N} V_{1n})(\sum_{n=1}^{N} V_{2n})\right) - \mu_1\mu_2(p\lambda T)^2.$$

Let $W_{1n} = \sum_{n=1}^{N} V_{1n}$ and $W_{2n} = \sum_{n=1}^{N} V_{2n}$. Now

$$E(W_{1n}W_{2n}) = \sum_{m=0}^{\infty} E(W_{1n}W_{2n} | N=m) \cdot Pr(n=m)$$

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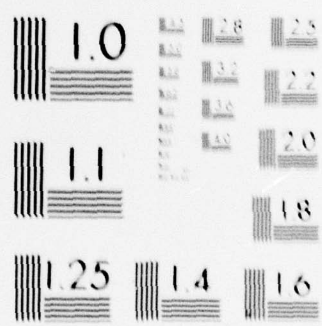

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$$= \sum_{m=0}^{\infty} E(W_{1N}|N=m)\ E(W_{2N}|N=m)\ \Pr(N=m)$$

$$= \sum_{m=0}^{\infty} (m\mu_1)(m\mu_2)\ ((p\lambda T)^m/m!)\ e^{-p\lambda T}$$

$$= \mu_1\mu_2 \sum_{m=0}^{\infty} m^2\ ((p\lambda T)^m/m!)\ e^{-p\lambda T}.$$

But $\operatorname{Var} N = p\lambda T = \sum_{m=0}^{\infty} m^2\ ((p\lambda T)^m/m!)\ e^{-p\lambda T} - \left\{\sum_{m=0}^{\infty} m((p\lambda T)^m/m!)\ e^{-p\lambda T}\right\}^2$

$$= \sum_{m=0}^{\infty} m^2\ ((p\lambda T)^m/m!)\ e^{-p\lambda T} - (p\lambda T)^2.$$

Hence,

$$E(W_{1N}W_{2N}) = \mu_1\mu_2((p\lambda T)^2+p\lambda T)$$

and

$$\operatorname{Cov}\left(\sum_{n=1}^{N} V_{1n},\ \sum_{n=1}^{N} V_{2n}\right) = \mu_1\mu_2 p\lambda T.$$

Q.E.D.

Lemma A6: MLSC′ is an unbiased estimator of RLSC′ ≡ RLSC - (NB)($\underline{\text{STK}}$)(UC), i.e., E(MLSC′) = RLSC′.

Proof: By IV.9,

$$E(MLSC') = \lambda(K_1p_3+K_2\mu_1+K_2p_1\mu_2+K_2p_2\mu_3+K_3p_3\mu_3+K_3K_4p_2\mu_4+K_3K_5p_3\mu_5).$$

RLSC′ ≡ RLSC - (NB)($\underline{\text{STK}}$)(UC)

= $C_1+C_2+C_3$-(NB)($\underline{\text{STK}}$)(UC) from Figure 1

$= \left[(\text{PFFH})(\text{UF})(\text{QPA})(1-\underline{\text{RIP}})(\underline{\text{NRTS}})/\underline{\text{MTBF}}\right](\text{DRCT})(\text{UC})$

$+\left[(\text{TFFH})(\text{UF})(\text{QPA})/\underline{\text{MTBF}}\right]\cdot\left[(\underline{\text{PAMH}}+(\underline{\text{RIP}})(\underline{\text{IMH}})+(1-\underline{\text{RIP}})(\underline{\text{RMH}}))\cdot\text{BLR}\right]$

$+\left[(\text{TFFH})(\text{UF})(\text{QPA})(1-\underline{\text{RIP}})/\underline{\text{MTBF}}\right].\left[(\underline{\text{RTS}})(\underline{\text{BMH}})(\text{BLR+BMR})+(\underline{\text{NRTS}})(\underline{\text{DMH}})(\text{DLR+DMR})\right]$

$= K_1(1-\underline{\text{RIP}})(\underline{\text{NRTS}})/\underline{\text{MTBF}}$

$+ K_2\left[\underline{\text{PAMH}}+(\underline{\text{RIP}})(\underline{\text{IMH}})+(1-\underline{\text{RIP}})(\underline{\text{RMH}})\right]/\underline{\text{MTBF}}$

$+ K_3\left[(1-\underline{\text{RIP}})/\underline{\text{MTBF}}\right].\left[K_4(\underline{\text{RTS}})(\underline{\text{BMH}})+K_5(\underline{\text{NRTS}})(\underline{\text{DMH}})\right]$ (by substituting III.5c)

$= K_1p_3\lambda+K_2\lambda\left[\mu_1+p_1\mu_2+(p_2+p_3)\mu_3\right]+K_3K_4\lambda p_2\mu_4+K_3K_5\lambda p_3\mu_5$

$= \lambda(K_1p_3+K_2\mu_1+K_2p_1\mu_2+K_2p_2\mu_3+K_2p_3\mu_3+K_3K_4p_2\mu_4+K_3K_5p_3\mu_5)$ (by substituting III.7 and IV.2)

= E(MLSC′).

Q.E.D.

APPENDIX B

DERIVATION OF PROPERTIES OF THE BASE SPARES COST TERM

Lemma B1: Given that $XBO(i,\gamma) = \sum_{x>i} (x-i)p(x|\gamma)$ where $p(x|\gamma)$ is Poisson with mean demand γ, then

$$XBO(i+1,\gamma) - XBO(i,\gamma) = \sum_{x=0}^{i} p(x|\gamma)-1 \ (<0),\ i\geq 0,\ \gamma>0.$$

Proof:

$$XBO(i+1,\gamma) - XBO(i,\gamma)$$

$$= \sum_{x>i+1} (x-i-1)p(x|\gamma) - \sum_{x>i} (x-i)p(x|\gamma)$$

$$= \sum_{x\geq i+1} (x-i-1)p(x|\gamma) - \sum_{x\geq i+1} (x-i)p(x|\gamma)$$

$$= \sum_{x\geq i+1} (x-i)p(x|\gamma) - \sum_{x\geq i+1} p(x|\gamma) - \sum_{x\geq i+1} (x-i)p(x|\gamma)$$

$$= -\sum_{x>i} p(x|\gamma) = -(1-\sum_{x=0}^{i} p(x|\gamma)) = \sum_{x=0}^{i} p(x|\gamma)-1 \ (<0), i\geq 0,\ \gamma>0.$$

Q.E.D.

Lemma B2: Given that $XBO(i,\gamma) = \sum_{x>i} (x-i)p(x|\gamma)$ where $p(x|\gamma)$ is Poisson with mean demand γ, then

a. $$\frac{\partial XBO(i,\gamma)}{\partial\gamma} = \sum_{x=i}^{\infty} p(x|\gamma),\ i\geq 0,\ \gamma>0, \text{ and}$$

b. $$\frac{\partial^2 XBO(i,\gamma)}{\partial\gamma^2} = p(i-1|\gamma),\ i\geq 1,\ \gamma>0.$$

Proof:

a. Given that $XBO(i,\gamma) = \sum_{x>i} (x-i)\, e^{-\gamma} \frac{\gamma^x}{x!} = e^{-\gamma} \sum_{x>i} (x-i)\frac{\gamma^x}{x!}$, then

$$\frac{\partial XBO(i,\gamma)}{\partial\gamma} = e^{-\gamma} \sum_{x>i} (x-i)\frac{x\gamma^{x-1}}{x!} - e^{-\gamma} \sum_{x>i} (x-i)\frac{\gamma^x}{x!}$$

$$= e^{-\gamma} \left[\sum_{x>i} (x-i) \frac{\gamma^{x-1}}{(x-1)!} - \sum_{x>i} (x-i)\frac{\gamma^x}{x!}\right]$$

$$= e^{-\gamma}\left[\sum_{y+1>i}(y+1-i)\frac{\gamma^y}{y!} - \sum_{x>i}(x-i)\frac{\gamma^x}{x!}\right] \quad \text{where } y \equiv x-1$$

$$= e^{-\gamma}\left[\sum_{y>i-1}(y-i)\frac{\gamma^y}{y!} + \sum_{y>i-1}\frac{\gamma^y}{y!} - \sum_{x>i}(x-i)\frac{\gamma^x}{x!}\right]$$

$$= e^{-\gamma}\left[\sum_{y>i}(y-i)\frac{\gamma^y}{y!} + \sum_{y=i}^{\infty}\frac{\gamma^y}{y!} - \sum_{x>i}(x-i)\frac{\gamma^x}{x!}\right]$$

$$= e^{-\gamma}\sum_{y=i}^{\infty}\frac{\gamma^y}{y!} = \sum_{y=i}^{\infty}e^{-\gamma}\frac{\gamma^y}{y!}$$

$$= \sum_{x=i}^{\infty}p(x|\gamma), \quad i\geq 0, \ \gamma>0.$$

Q.E.D.

b. Given that $\frac{\partial XBO(i,\gamma)}{\partial\gamma} = \sum_{x=i}^{\infty}p(x|\gamma) = \sum_{x=i}^{\infty}e^{-\gamma}\frac{\gamma^x}{x!}$, then

$$\frac{\partial^2 XBO(i,\gamma)}{\partial\gamma^2} = \frac{\partial}{\partial\gamma}\left(e^{-\gamma}\sum_{x=i}^{\infty}\frac{\gamma^x}{x!}\right)$$

$$= e^{-\gamma}\sum_{x=i}^{\infty}\frac{x\gamma^{x-1}}{x!} - e^{-\gamma}\sum_{x=i}^{\infty}\frac{\gamma^x}{x!}$$

$$= e^{-\gamma}\left[\sum_{x=i}^{\infty}\frac{\gamma^{x-1}}{(x-1)!} - \sum_{x=i}^{\infty}\frac{\gamma^x}{x!}\right]$$

$$= e^{-\gamma}\left[\sum_{y=i-1}^{\infty}\frac{\gamma^y}{y!} - \sum_{x=i}^{\infty}\frac{\gamma^x}{x!}\right] \quad \text{where } y\equiv x-1$$

$$= e^{-\gamma}\frac{\gamma^{i-1}}{(i-1)!} = p(i-1|\gamma), \quad i\geq 1, \ \gamma>0.$$

Q.E.D.

Lemma B3: Given that

$a_i(EBO) \equiv$ the largest value of γ such that $STK(\gamma,EBO)=i$, $i=0,1,2,\ldots,$

then $a_{i+1}(EBO) - a_i(EBO) < 1$.

Proof: Recall from V.3 and Lemma B1 that

$$XBO(i+1,\gamma) - XBO(i,\gamma) = -\sum_{x=i+1}^{\infty}p(x|\gamma), \quad i\geq 0, \ \gamma>0.$$

Also, if $\gamma_2>\gamma_1>0$, we have by Lemma B2 that

$$XBO(i+1,\gamma_2) > XBO(i+1,\gamma_1) + \frac{\partial XBO(i+1,\gamma_1)}{\partial\gamma_1}(\gamma_2-\gamma_1),$$

or, in particular,

$$XBO(i+1,\gamma+1) > XBO(i+1,\gamma) + \frac{\partial XBO(i+1,\gamma)}{\partial\gamma} \times 1 = XBO(i+1,\gamma) + \sum_{x=i+1}^{\infty} p(x|\gamma)$$

Combining these results, we have

$$XBO(i+1,\gamma+1)-XBO(i,\gamma)$$

$$= \underbrace{\left[XBO(i+1,\gamma+1) - XBO(i+1,\gamma)\right]}_{> \sum_{x=i+1}^{\infty} p(x|\gamma)} + \underbrace{\left[XBO(i+1,\gamma) - XBO(i,\gamma)\right]}_{= -\sum_{x=i+1}^{\infty} p(x|\gamma)} > 0$$

In particular,

$$XBO(i+1,a_i(EBO)+1)>XBO(i,a_i(EBO))$$

But, by the definition of $a_i(EBO)$, $i=0,1,\ldots,$

$XBO(i,a_i(EBO)) = XBO(i+1,a_{i+1}(EBO)) = \ldots.=EBO.$ Hence,

$$XBO(i+1,a_i(EBO)+1)>XBO(i+1,a_{i+1}(EBO))$$

which implies that

$$a_{i+1}(EBO)<a_i(EBO)+1 \text{ or}$$

$$a_{i+1}(EBO)-a_i(EBO)<1, \; 0<EBO, \; i=1,2,\ldots$$

Q.E.D.

<u>Lemma B4</u>: If $P(X_2(T)=m_2) = \frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}$,

$P(X_3(T)=m_3)= \frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}$, and $\tilde{\gamma} = K_6X_2(T)+K_7X_3(T)$,

then

$$Pr(a_{i-1}(EBO)<\tilde{\gamma}\le a_i(EBO)) = \sum_{\substack{m_2,m_3 \\ a_{i-1}(EBO)<K_6m_2+K_7m_3\le a_i(EBO)}} \left[\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right]\left[\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right].$$

Proof:

$$\Pr(a_{i-1}(EBO)<\tilde{\gamma}\leq a_i(EBO)) = \Pr(a_{i-1}(EBO)<K_6X_2(T)+K_7X_3(T)\leq a_i(EBO))$$

$$= \sum_{\substack{m_2,m_3,\\ a_{i-1}(EBO)<K_6m_2+K_7m_3\leq a_i(EBO)}} \Pr(X_2(T)=m_2, X_3(T)=m_3)$$

$$= \sum_{\substack{m_2,m_3,\\ a_{i-1}(EBO)<K_6m_2+K_7m_3\leq a_i(EBO)}} \Pr(X_2(T)=m_2)\Pr(X_3(T)=m_3) \quad \text{by Lemma A1}$$

$$= \sum_{\substack{m_2,m_3,\\ a_{i-1}(EBO)<K_6m_2+K_7m_3\leq a_i(EBO)}} \left[\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right]\left[\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right].$$

Q.E.D.

Lemma B5: If $P(X_2(T)=m_2) = \frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}$;

$P(X_3(T)=m_3) = \frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}$; $\tilde{\gamma} = K_6X_2(T) + K_7X_3(T)$;

and $\widetilde{STK}(\tilde{\gamma}, EBO)=i$, $a_{i-1}(EBO)<\tilde{\gamma}\leq a_i(EBO)$, $i=0,1,\ldots$;

then

$$\Pr(\widetilde{STK}(\tilde{\gamma},EBO)=i,\ X_3(T)=j) = \begin{cases} \dfrac{(p_3\lambda T)^j}{j!} e^{-p_3\lambda T}\left[\displaystyle\sum_{\substack{m,\\ \max(0,\frac{a_{i-1}(EBO)-K_7j}{K_6})<m\leq\frac{a_i(EBO)-K_7j}{K_6}}} \frac{(p_2\lambda T)^m}{m!} e^{-p_2\lambda T}\right] \\ \qquad i\geq 1,\ 0\leq j\leq M_i^1 \equiv [a_i(EBO)/K_7] \\ 0, \text{ otherwise.} \end{cases}$$

Proof:

$$Pr(S\widetilde{T}K(\tilde{\gamma},EBO) = i,\ X_3(T)=j) = Pr(a_{i-1}(EBO)<\tilde{\gamma}\le a_i(EBO),\ X_3(T)=j)$$

$$= Pr(a_{i-1}(EBO)<K_6X_2(T)+K_7X_3(T)\le a_i(EBO),\ X_3(T)=j)$$

$$= Pr(a_{i-1}(EBO)<K_6X_2(T)+K_7X_3(T)\le a_i(EBO)\mid X_3(T)=j)\ Pr(X_3(T)=j)$$

$$= Pr(a_{i-1}(EBO)<K_6X_2(T)+K_7j\le a_i(EBO))\ Pr(X_3(T)=j)$$

$$= Pr(\max(0,a_{i-1}(EBO)-K_7j)<K_6X_2(T)\le a_i(EBO)-K_7j)\ Pr(X_3(T)=j)$$

$$= Pr(\max(0,\frac{a_{i-1}(EBO)-K_7j}{K_6})<X_2(T)\le\frac{a_i(EBO)-K_7j}{K_6})\ Pr(X_3(T)=j)$$

$$= \frac{(p_3\lambda T)^j}{j!}e^{-p_3\lambda T}\left[\sum_{\substack{m\ni\\ \max(0,\frac{a_{i-1}(EBO)-K_7j}{K_6})<m\le\frac{a_i(EBO)-K_7j}{K_6}}}\frac{(p_2\lambda T)^m}{m!}e^{-p_2\lambda T}\right]$$

But the above summation in brackets is undefined if

$$\frac{a_i(EBO)-K_7j}{K_6} < 0,$$

or $a_i(EBO)-K_7j<0$,

or $j>a_i(EBO)/K_7$.

Hence, we have

$$Pr(S\widetilde{T}K(\tilde{\gamma},EBO)=i,X_2(T)=j) = \begin{cases} \dfrac{(p_3\lambda T)^j}{j!}e^{-p_3\lambda T}\displaystyle\sum_{\substack{m\ni\\ \max(0,\frac{a_{i-1}(EBO)-K_7j)}{K_6})<m\le\frac{a_i(EBO)-K_7j}{K_6}}}\frac{(p_2\lambda T)^m}{m!}e^{-p_2\lambda T} \\ \qquad i\ge1,\ 0\le j\le M_i^1 \equiv [a_i(EBO)/K_7] \\ 0, \text{ otherwise.}\end{cases}$$

Q.E.D.

Lemma B6: Given that $P(X_2(T) = m_2) = \dfrac{(p_2\lambda T)^{m_2}}{m_2!}e^{-p_2\lambda T}$;

$$P(X_3(T)=m_3) = \frac{(p_3\lambda T)^{m_3}}{m_3}e^{-p_3\lambda T};\ \tilde{\gamma}=K_6X_2(T)+K_7X_3(T);\ S\widetilde{T}K(\tilde{\gamma},EBO)=i,$$

$a_{i-1}(EBO)<\tilde{\gamma}\le a_i(EBO), i=1,\ldots$; and $\sum_{k=0}^{X_2(T)} Y_{4k}$ is a compound Poisson sum with $E(Y_{4k}) = \mu_4$; then

$$E(S\widetilde{T}K(\tilde{\gamma},EBO) \sum_{k=0}^{X_2(T)} Y_{4k}) = \mu_4 \sum_{i=0}^{\infty} i \sum_{j=0}^{M_i^2} \frac{(p_2\lambda T)^j}{j!} e^{-p_2\lambda T} \left[\sum_{\max(0,\frac{a_{i-1}(EBO)-K_6 j}{K_7})<m\le\frac{a_i(EBO)-K_6 j}{K_7}} \frac{(p_3\lambda T)^m}{m!} e^{-p_3\lambda T} \right]$$

$$\text{(where } M_i^2 = [a_i(EBO)/K_6])$$

$$= \mu_4 E(S\widetilde{T}K(\tilde{\gamma},EBO)X_2(T))$$

<u>Proof</u>:

$$E(S\widetilde{T}K(\tilde{\gamma},EBO) \cdot \sum_{k=0}^{X_2(T)} Y_{4k})$$

$$= \sum_{m=0}^{\infty} E(S\widetilde{T}K(\tilde{\gamma},EBO) \cdot \sum_{k=0}^{m} Y_{4k} | X_2(T)=m) \cdot P(X_2(T)=m)$$

$$= \sum_{m=0}^{\infty} E(S\widetilde{T}K(\tilde{\gamma},EBO) | X_2(T)=m) \cdot E(\sum_{k=0}^{m} Y_{4k} | X_2(T)=m) \cdot P(X_2(T)=m)$$

$$= \sum_{m=0}^{\infty} \{\sum_i i\ P(S\widetilde{T}K(\tilde{\gamma},EBO)=i | X_2(T)=m)\}\ m\mu_4 \frac{(p_2\lambda T)^m}{m!} e^{-p_2\lambda T}$$

$$= \sum_{i=0}^{\infty} i \sum_{m=0}^{M_i^2} P(\frac{a_{i-1}(EBO)-K_6 m}{K_7} < X_3(T) \le \frac{a_i(EBO)-K_6 m)}{K_7}) \cdot m\mu_4 e^{-p_2\lambda T} \frac{(p_2\lambda T)^m}{m!}$$

$$= \mu_4 \sum_{i=0}^{\infty} i \sum_{m=0}^{M_i^2} m\ e^{-p_2\lambda T} \frac{(p_2\lambda T)^m}{m!} \left\{ \sum_{\max\left(0,\frac{a_{i-1}(EBO)-K_6 m}{K_7}\right)<j\le\frac{a_i(EBO)-K_6 m}{K_7}} e^{-p_3\lambda T} (p_3\lambda T)^j/j! \right\}$$

$$= \mu_4 E(S\widetilde{T}K(\tilde{\gamma},EBO)X_2(T)), \text{ by V.25.}$$

Q.E.D.

Lemma B7:

$$E(\widetilde{STK}^n(\tilde{\gamma},EBO)) = \sum_{i=0}^{\infty} i^n \sum_{\substack{m_2,m_3, \\ a_{i-1}(EBO)<K_6m_2+K_7m_3\leq a_i(EBO)}} \left[\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right]\left[\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right] < \infty$$

Proof:*

For notational convenience let $\alpha = p_2\lambda T$, $\beta = p_3\lambda T$, and $A_1 \equiv E(\widetilde{STK}^n(\tilde{\gamma},EBO))$. Hence

$$A_1 = \sum_{i=0}^{\infty} i^n \sum_{\substack{m_2,m_3, \\ a_i<K_6m_2+K_7m_3\leq a_{i+1}}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!}$$

(where a_i is defined the same as a_{i-1} was defined in V.31)

Since $a_{i+1} < a_i+1$ by Lemma B3, we know that

$$A_1 \leq \sum_{i=0}^{\infty} i^n \sum_{a_i<K_6m_2+K_7m_3\leq a_i+1} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!} \equiv A_2$$

Without loss of generality, assume $K_6 < K_7$. As shown by Figure B1,

$$A_2 \leq \sum_{i=0}^{\infty} i^n \sum_{[\tilde{\alpha}_i]+1\leq m_2+m_3\leq[\tilde{\delta}_i]} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!} \equiv A_3,$$

where $\tilde{\alpha}_i = a_i/K_7$ and $\tilde{\delta}_i = (\alpha_i + 1)/K_6$.

Now let $M = X_2(T)+X_3(T)$. We known that M is Poisson with parameter $\delta = \alpha+\beta$ [22, p. 71]. Let $\phi_M(m)$, m=0, 1, 2, ..., be the probability mass function (pmf) of m. Then

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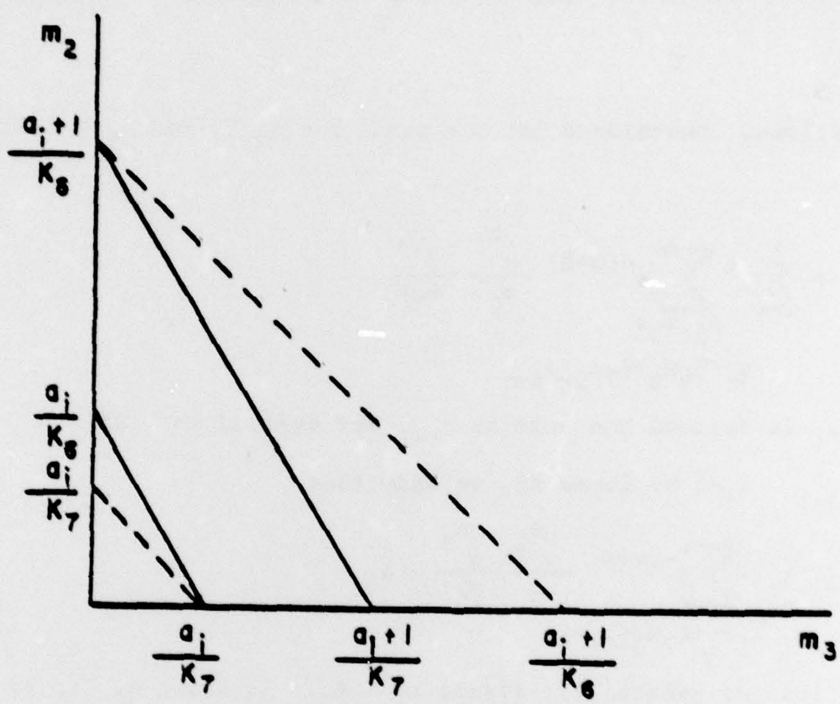
Figure B-1. **ILLUSTRATION OF RELATIVE PROBABILITIES**



$$A_3 = \sum_{i=0}^{\infty} i^n \sum_{[\tilde{a}_i]+1 \le m \le [\tilde{b}_i]} \phi_M(m)$$

But $\phi_M(m)$ decreases as m increases for $m>\delta$ (since $\phi_M(m) = \frac{\delta}{m}\phi_M(m-1)$ [22, p.1]). Moreover, since $a_i \to \infty$ as $i \to \infty$, there exists an $i=n \ni [\tilde{\alpha}_n]+1 \le \delta$ and $[\tilde{\alpha}_i]+1>\delta$ for $i>n$. By this, we have

$$A_3 \le \underbrace{\sum_{i=0}^{n} i^n \sum_{[\tilde{a}_i]+1 \le m \le [\tilde{b}_i]} \phi_M(m)}_{A_4<\infty} + \underbrace{\sum_{i=n+1}^{\infty} i^n \sum_{[\tilde{a}_i]+1 \le m \le [\tilde{b}_i]} \phi_M([\tilde{\alpha}_i]+1)}_{A_5},$$

and clearly,

$$A_5 \le \sum_{i=n+1}^{\infty} i^n \sum_{m \le [\tilde{b}_i]} \phi_M([\tilde{a}_i]+1) \equiv A_6.$$

But $a_o = 0$, $a_1 = EBO$, $a_i - a_{i-1} < 1$ for $i \geq 2$ implies that $a_i < EBO+(i-1)$.

Hence $[\tilde{a}_i] = [\frac{a_i+1}{K_6}] \leq \frac{a_i+1}{K_6} < \frac{1}{K_6}(EBO+i)$ so that

$$A_6 \leq \frac{1}{K_6} \sum_{i=n+1}^{\infty} i^n(i+EBO)\phi_M([\tilde{a}_i]+1) \equiv A_7.$$

But $A_7 = \frac{1}{K_6} \sum_{k=s}^{\infty} u_k$ where $u_k = \sum_{i=i_k}^{i_k+j_k-1} i^n(i+EBO)\phi_M([\tilde{a}_i]+1)$, and the limits on the u_k sum are selected such that $[\tilde{a}_i]+1=k$ for all $i=i_k,\ldots,i_k+j_k-1$, and s is that value of k for which $i_k = n+1$. In other words, j_k is the number of values of i for which $[\tilde{a}_i]+1$ takes on the value k, and i_k is the smallest value of i for which this occurs (so that $i_k = \sum_{r=1}^{k-1} j_r+1$). Hence, $u_k = \phi_M(k) \sum_{i=i_k}^{i_k+j_k} i^n(i+EBO)$

so that

$$A_7 = \frac{1}{K_6} \sum_{k=s}^{\infty} \phi_M(k) \sum_{i=i_k}^{i_k+j_k-1} i^n(i+EBO).$$

Now let $J_k = \max\limits_{r=1,2,\ldots,k} (j_r)$. $\{J_k\}$ is a nondecreasing sequence of positive integers. Given that we keep EBO constant, then as $\gamma \to \infty$, the stock level must increase by 1 for each unit increase in γ. This implies that $j_k \to 1$ as $\gamma \to \infty$. Thus there exists a finite upper bound on the J_k, say J, since j_k is finite for all k and $j_k \to 1$ as $k \to \infty$.

$$\text{Hence, } A_7 \leq \frac{1}{K_6} \sum_{k=s}^{\infty} \phi_M(k) \underbrace{\sum_{i=(k-1)J+1}^{kJ} i^n(i+EBO)}_{B_k}.$$

Using the ratio test of convergence [23, p. 57] and the fact that the limit of a product of partial sums is the product of the limits of partial sums, we have,

$$B_{k+1}/B_k = \left(\sum_{i=kJ+1}^{(k+1)J} i^n(i+EBO)\right)/\left(\sum_{i=(k-1)J+1}^{kJ} i^n(i+EBO)\right)$$

$$= \frac{(kJ+1)^n(kJ+1+EBO)+(kJ+2)^n(kJ+2+EBO)+..+(kJ+J)^n(kJ+J+EBO)}{((k-1)J+1)^n((k-1)J+1+EBO)+((k-1)J+2)^n((k-1)J+2+EBO)+..+((k-1)J+J)^n((k-1)J+J+EBO)}$$

$$= \underbrace{(k^{n+1}/(k-1)^{n+1})}_{C_k} \underbrace{\left\{\begin{array}{l} [(J+1/k)^n(J+1/k+EBO/k)+(J+2/k)^n(J+2/k+EBO/k)+.. \\ +(J+J/k)^n(J+J/k+EBO/k)]/[(J+1/(k-1))^n(J+1/(k-1)+EBO/(k-1)) \\ +(J+2/(k-1))^n(J+2/(k-1)+EBO/(k-1))+.. \\ +(J+J/(k-1))^n(J+J/(k-1)+EBO/(k-1))] \end{array}\right\}}_{D_k}$$

and $\lim_{k\to\infty} C_k = 1$, $\lim_{k\to\infty} D_k = 1$, and $\lim_{k\to\infty} (\phi_M(k+1)/\phi_M(k))$

$= \lim_{k\to\infty} (\frac{\delta}{(k+1)} \phi_M(k)/\phi_M(k)) = \lim_{k\to\infty} (\delta/(k+1)) = 0$ so that

$$\lim_{k\to\infty} \frac{\phi_M(k+1)B_{k+1}}{\phi_M(k)B_k} = (\lim_{k\to\infty} (\phi_M(k+1)/\phi_M(k)))(\lim_{k\to\infty} C_k)(\lim_{k\to\infty} D_k) = 0.$$

Hence $A_7<\infty$ and therefore $A_1<\infty$. Q.E.D.

Lemma B8 (see V.24):

$$E(S\widetilde{T}K(\tilde{\gamma},EBO)X_3(T)) < \infty$$

Proof:

$$E(S\widetilde{T}K(\tilde{\gamma},EBO)X_3(T)) = \sum_{i=0}^{\infty} i \sum_{a_{i-1}(EBO)<K_6m_2+K_7j\le a_{i+1}(EBO)} j\left\{\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right\}\left\{\frac{(p_3\lambda T)^j}{j!} e^{-p_3\lambda T}\right\}$$

$$=\sum_{i=0}^{\infty} i \sum_{a_i<K_6m_2+K_7j\le a_{i+1}} j e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^j}{j!} \equiv E_1,$$ where, as in Lemma B7, we let

a_i be defined as a_{i-1} was defined in V.31, $\alpha=p_2\lambda T$, and $\beta=p_3\lambda T$. But

$$E_1 = \sum_{i=0}^{\infty} i\beta \sum_{a_i < K_6 m_2 + K_7 j \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{j-1}}{(j-1)!}$$

$$= \beta \sum_{i=0}^{\infty} i \sum_{a_i < K_6 m_2 + K_7 (x_3+1) \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{x_3}}{x_3!} \qquad \text{where } x_3 \equiv j-1,$$

$$= \beta \sum_{i=0}^{\infty} i \sum_{(a_i - K_7) < K_6 m_2 + K_7 x_3 \le (a_{i+1} - K_7)} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{x_3}}{x_3!}$$

But E_1 has the same form as A_1 in the proof of Lemma B7 and hence, the remainder of this proof completely parallels the Lemma B7 proof.

Lemma B9:

a. $\partial E(\widetilde{STK}(\tilde{\gamma}, EBO))/\partial\lambda$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1} < K_6 m_2 + K_7 m_3 \le a_i} \left(\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right)\left(\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right)\lambda^{-1}((m_2+m_3) - \lambda T(p_2+p_3))$$

b. $\partial^2 E((\widetilde{STK}(\tilde{\gamma}, EBO))/\partial\lambda^2$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1} < K_6 m_2 + K_7 m_3 \le a_i} \left(\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right)\left(\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right)\lambda^{-2}[((m_2+m_3) - \lambda T(p_2+p_3))^2$$

$$-(m_2+m_3)]$$

<u>Proof</u>: As we have seen,

$E(S\widetilde{T}K(\tilde{\gamma}, EBO))$

$$= \sum_{i=0}^{\infty} i \left\{ \sum_{a_{i-1}<K_6 m_2+K_7 m_3 \le a_i} \left[\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right]\left[\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right] \right\}. \text{ Let } g = p_2^{m_2} p_3^{m_3} T^{m_2+m_3}/(m_2!m_3!).$$

Then $\partial E(S\widetilde{T}K(\tilde{\gamma}, EBO))/\partial\lambda$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1}<K_6 m_2+K_7 m_3 \le a_i} g[e^{-\lambda T(p_2+p_3)}(m_2+m_3)\lambda^{m_2+m_3-1} - T(p_2+p_3)\lambda^{m_2+m_3} e^{-\lambda T(p_2+p_3)}]$$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1}<K_6 m_2+K_7 m_3 \le a_i} g\, \lambda^{m_2+m_3-1} e^{-\lambda T(p_2+p_3)}((m_2+m_3)-\lambda T(p_2+p_3))$$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1}<K_6 m_2+K_7 m_3 \le a_i} \left(\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right)\left(\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right)\lambda^{-1}((m_2+m_3)-\lambda T(p_2+p_3)) \quad \text{Q.E.D.}$$

Let $F = \sum_{i=0}^{\infty} i \left\{ \sum_{a_{i-1}<K_6 m_2+K_7 m_3 \le a_i} g\, \lambda^{m_2+m_3-1} e^{-\lambda T(p_2+p_3)}((m_2+m_3)-\lambda T(p_2+p_3)) \right\}$. Then

$\partial^2 E(S\widetilde{T}K(\tilde{\gamma}, EBO))/\partial\lambda^2 = \partial F/\partial\lambda$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1}<K_6 m_2+K_7 m_3 \le a_i} g \cdot [((m_2+m_3)-\lambda T(p_2+p_3))(e^{-\lambda T(p_2+p_3)}(m_2+m_3-1)\lambda^{m_2+m_3-2}$$

$$-T(p_2+p_3)e^{-\lambda T(p_2+p_3)}\lambda^{m_2+m_3-1}) + \lambda^{m_2+m_3-1} e^{-\lambda T(p_2+p_3)}(-T(p_2+p_3))]$$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1}<K_6m_2+K_7m_3\le a_i} g\lambda^{m_2+m_3-2} e^{-\lambda T(p_2+p_3)} [((m_2+m_3)-\lambda T(p_2+p_3))((m_2+m_3-1)$$

$$-\lambda T(p_2+p_3))-\lambda T(p_2+p_3)]$$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1}<K_6m_2+K_7m_3\le a_i} g\lambda^{m_2+m_3-2} e^{-\lambda T(p_2+p_3)} [((m_2+m_3)-\lambda T(p_2+p_3))^2-\lambda T(p_2+p_3)$$

$$-(m_2+m_3)+\lambda T(p_2+p_3)]$$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1}<K_6m_2+K_7m_3\le a_i} g\lambda^{m_2+m_3-2} e^{-\lambda T(p_2+p_3)} [((m_2+m_3)-\lambda T(p_2+p_3))^2-(m_2+m_3)]$$

$$= \sum_{i=0}^{\infty} i \sum_{a_{i-1}<K_6m_2+K_7m_3\le a_i} \left(\frac{(p_2\lambda T)^{m_2}}{m_2!} e^{-p_2\lambda T}\right)\left(\frac{(p_3\lambda T)^{m_3}}{m_3!} e^{-p_3\lambda T}\right)\lambda^{-2}[((m_2+m_3)-\lambda T(p_2+p_3))^2$$

$$-(m_2+m_3)]$$

Q.E.D.

<u>Lemma B10</u>: Given that

$\partial E(S\widetilde{T}K(\tilde{\gamma},EBO))/\partial\lambda$

$$= \sum_{i=0}^{\infty} (i/\lambda) \sum_{a_i<K_6m_2+K_7m_3\le a_{i+1}} e^{-\lambda(p_2+p_3)} \frac{(p_2\lambda T)^{m_2}}{m_2!} \frac{(p_3\lambda T)^{m_3}}{m_3!} ((m_2+m_3)-\lambda T(p_2+p_3)), \text{ then}$$

$0 \le \partial E(S\widetilde{T}K(\tilde{\gamma},EBO))/\partial\lambda < (p_2+p_3)T$ if $a_{i+1} - a_i > K_6, K_7$ for all i.

Proof: From Lemma B9, we know that

$\partial E(S\widetilde{T}K(\tilde{\gamma},EBO))/\partial\lambda$

$$= \sum_{i=0}^{\infty} (i/\lambda) \sum_{a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!} ((m_2+m_3)-(\alpha+\beta)), \text{ where } \alpha = p_2\lambda T,\ \beta = p_3\lambda T,$$

from which we see that

$\partial E(S\widetilde{T}K(\tilde{\gamma}, EBO))/\partial\lambda$

$$= \sum_{i=0}^{\infty} (i/\lambda) \left\{ \sum_{a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1}} (m_2+m_3) e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!} - (\alpha+\beta) \sum_{a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!} \right\}$$

$$= \sum_{i=0}^{\infty} (i/\lambda) \left\{ \alpha \sum_{a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2-1}}{(m_2-1)!} \frac{\beta^{m_3}}{m_3!} + \beta \sum_{a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3-1}}{(m_3-1)!} \right.$$

$$\left. -(\alpha+\beta) \sum_{a_i < K_6 m_2 + K_7 m_3 \leq^2 a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!} \right\}$$

$$= \sum_{i=0}^{\infty} (i/\lambda) \left\{ \alpha \left[\sum_{a_i - K_6 < K_6 m_2' + K_7 m_3 \leq a_{i+1} - K_6} e^{-(\alpha+\beta)} \frac{\alpha^{m_2'}}{m_2'!} \frac{\beta^{m_3}}{m_3!} - \sum_{a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!} \right] \right.$$

$$\left. + \beta \left[\sum_{a_i - K_7 < K_6 m_2 + K_7 m_3' \leq a_{i+1} - K_7} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3'}}{m_3'!} - \sum_{a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_2!} \right] \right\}$$

where $m_2 - 1 = m_2'$ or $m_2 = m_2' + 1$, and hence $(a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1})$ if and only if $(a_i - K_6 < K_6 m_2' + K_7 m_3 \leq a_{i+1} - K_6)$. In a similar fashion, $(a_i < K_6 m_2 + K_7 m_3 \leq a_{i+1})$ if and only if $(a_i - K_7 < K_6 m_2 + K_7 m_3' \leq a_{i+1} - K_7)$ in the case of the term multiplied by β.

If $a_{i+1} - a_i > K_6, K_7$, then

$a_i - K_j < a_i < a_{i+1} - K_j < a_{i+1}$, $j = 6, 7$,

i.e., the two bands of probability in each $[\cdot]$ expression overlap. Thus,

$\partial E(S\widetilde{T}K(\tilde{\gamma},\ EBO))/\partial\lambda$

$$= \sum_{i=0}^{\infty} (i/\lambda)\Bigg\{\alpha\Bigg[\sum_{a_i - K_6 < K_6 m_2 + K_7 m_3 \le a_i} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!} - \sum_{a_{i+1} - K_6 < K_6 m_2 + K_7 m_3 \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!}\Bigg]$$

$$+\beta\Bigg[\sum_{a_i - K_7 < K_6 m_2 + K_7 m_3 \le a_i} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!} - \sum_{a_{i+1} - K_7 < K_6 m_2 + K_7 m_3 \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_2!}\Bigg]\Bigg\}$$

since $a_{i+1} - a_i > K_6,\ K_7$ for all i.

Consequently,

$\partial E(S\widetilde{T}K(\tilde{\gamma},\ EBO))/\partial\lambda$

$$= (\alpha/\lambda)\left\{\sum_{i=0}^{\infty} i \sum_{a_i - K_6 < K_6 m_2 + K_7 m_3 \le a_i} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!} - \sum_{i=0}^{\infty} i \sum_{a_{i+1} - K_6 < K_6 m_2 + K_7 m_3 \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!}\right\}$$

$$+ (\beta/\lambda)\left\{\sum_{i=0}^{\infty} i \sum_{a_i - K_7 < K_6 m_2 + K_7 m_3 \le a_i} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!} - \sum_{i=0}^{\infty} i \sum_{a_{i+1} - K_7 < K_6 m_2 + K_7 m_3 \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!}\right\}$$

$$= (\alpha/\lambda)\Bigg\{\sum_{i=0}^{\infty} i \sum_{a_i - K_6 < K_6 m_2 + K_7 m_3 \le a_i} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!} - \sum_{i=0}^{\infty} (i+1) \sum_{a_{i+1} - K_6 < K_6 m_2 + K_7 m_3 \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!}$$

$$+ \sum_{i=0}^{\infty} \sum_{a_{i+1} - K_6 < K_6 m_2 + K_7 m_3 \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!}\Bigg\}$$

$$+ (\beta/\lambda)\Bigg\{\sum_{i=0}^{\infty} i \sum_{a_i - K_7 < K_6 m_2 + K_7 m_3 \le a_i} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!} - \sum_{i=0}^{\infty} (i+1) \sum_{a_{i+1} - K_7 < K_6 m_2 + K_7 m_3 \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!}$$

$$+ \sum_{i=0}^{\infty} \sum_{a_{i+1} - K_7 < K_6 m_2 + K_7 m_3 \le a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!}\frac{\beta^{m_3}}{m_3!}\Bigg\}$$

But since the $i=0$ term is 0 in the first of the three summations in each case, the first and second summations cancel. Hence,

$\partial E(\widetilde{STK}(\tilde{\gamma}, EBO))/\partial\lambda$

$$= (\alpha/\lambda) \left\{\sum_{i=0}^{\infty} \sum_{a_{i+1}-K_6<K_6m_2+K_7m_3\leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!}\right\} + (\beta/\lambda) \left\{\sum_{i=0}^{\infty} \sum_{a_{i+1}-K_7<K_6m_2+K_7m_3\leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!}\right\}$$

≥ 0 whenever $a_{i+1}-a_i > K_6, K_7$ for all i.

Furthermore, since $a_{i+1}-a_i > K_6, K_7$,

$\partial E(\widetilde{STK}(\tilde{\gamma}, EBO))/\partial\lambda$

$$\leq (\alpha/\lambda) \left\{\sum_{i=0}^{\infty} \sum_{a_i<K_6m_2+K_7m_3\leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!}\right\} + (\beta/\lambda) \left\{\sum_{i=0}^{\infty} \sum_{a_i<K_6m_2+K_7m_3\leq a_{i+1}} e^{-(\alpha+\beta)} \frac{\alpha^{m_2}}{m_2!} \frac{\beta^{m_3}}{m_3!}\right\}$$

$$= (\alpha/\lambda) \sum_{i=0}^{\infty} Pr(a_i<K_6m_2+K_7m_3\leq a_{i+1}) + (\beta/\lambda) \sum_{i=0}^{\infty} Pr(a_i<K_6m_2+K_7m_3\leq a_{i+1})$$

$$= (\alpha+\beta)/\lambda = (p_2\lambda T+p_3\lambda T)/\lambda = (p_2+p_3)T \qquad \text{Q.E.D.}$$

(Note from the example values of $(a_{i+1}-a_i)$ in Figure 5 and the example values of K_6 and K_7 in Figure 6 that the conditions, $a_{i+1}-a_i>K_6$, K_7 appear to be met for variable and parameter values used to date (EBO = .1) in the LSCC structuring environment.)

<u>Lemma B11</u>:

$$\begin{aligned} Var(MLSC) = {} & VAR(MLSC') + K_0^2\ Var\ (\widetilde{STK}(\tilde{\gamma}, EBO)) \\ & + (2/T)(K_0K_2\mu_1+K_0K_2\mu_3+K_0K_3K_4\mu_4)Cov(\widetilde{STK}(\tilde{\gamma}, EBO), X_2(T)) \\ & + (2/T)(K_0K_1+K_0K_2\mu_1+K_0K_2\mu_3+K_0K_3K_5\mu_5)Cov(\widetilde{STK}(\tilde{\gamma}, EBO), X_3(T)), \end{aligned}$$

where Var(MLSC´) is given by IV.10a, $Var(\widetilde{STK}(\tilde{\gamma}, EBO))$ by V.21, $Cov(\widetilde{STK}(\tilde{\gamma}, EBO), X_2(T))$ by V.29, and $Cov(\widetilde{STK}(\tilde{\gamma}, EBO), X_3(T))$ by V.30.

<u>Proof</u>: The variance of V.33 is

$$\mathrm{Var(MLSC)} = K_0^2\ \mathrm{Var}(\widetilde{STK}(\tilde{\gamma},\ EBO)) + \mathrm{Var}\ (MLSC') + 2(K_0K_1/T)\mathrm{Cov}(\widetilde{STK}(\tilde{\gamma},\ EBO),\ X_3(T))$$

$$+\ 2(K_0K_2/T)\mathrm{Cov}(\widetilde{STK}(\tilde{\gamma},\ EBO), \sum_{k=1}^{X_2(T)} Y_{1k}^{(2)})$$

$$+\ 2(K_0K_2/T)\mathrm{Cov}(\widetilde{STK}(\tilde{\gamma},\ EBO), \sum_{k=1}^{X_2(T)} Y_{3k}^{(2)})$$

$$+\ 2(K_0K_3K_4/T)\mathrm{Cov}(\widetilde{STK}(\tilde{\gamma},\ EBO), \sum_{k=1}^{X_2(T)} Y_{4k})$$

$$+\ 2(K_0K_2/T)\mathrm{Cov}(\widetilde{STK}(\tilde{\gamma},\ EBO), \sum_{k=1}^{X_3(T)} Y_{1k}^{(3)})$$

$$+\ 2(K_0K_2/T)\mathrm{Cov}(\widetilde{STK}(\tilde{\gamma},\ EBO), \sum_{k=1}^{X_3(T)} Y_{3k}^{(3)})$$

$$+\ 2(K_0K_3K_5/T)\mathrm{Cov}\ (\widetilde{STK}(\tilde{\gamma},\ EBO), \sum_{k=1}^{X_3(T)} Y_{5k}).$$

Expanding the covariance terms and substituting from V.27, V.28, IV.5 and IV.7a leads to

$$\mathrm{Var(MLSC)} = K_0^2\ \mathrm{Var}(\widetilde{STK}(\tilde{\gamma},\ EBO))+\mathrm{Var}(MLSC')$$

$$+\ 2(K_0K_1/T)(E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_3(T))-p_3\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO))$$

$$+\ 2(K_0K_2/T)(\mu_1E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_2(T))-\mu_1p_2\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO)))$$

$$+\ 2(K_0K_2/T)(\mu_3E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_2(T))-\mu_3p_2\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO)))$$

$$+\ 2(K_0K_3K_4/T)(\mu_4E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_2(T))-\mu_4p_2\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO)))$$

$$+\ 2(K_0K_2/T)(\mu_1E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_3(T))-\mu_1p_2\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO)))$$

$$+\ 2(K_0K_2/T)(\mu_3E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_3(T))-\mu_3p_3\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO)))$$

$$+\ 2(K_0K_3K_5/T)(\mu_5E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_3(T))-\mu_5p_3\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO)))$$

Regrouping terms, we have

$$\mathrm{Var(MLSC)} = K_0^2\ \mathrm{Var}(\widetilde{STK}(\tilde{\gamma},\ EBO)) + \mathrm{Var}(MLSC')$$

$$+\ 2(E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_2(T)) - p_2\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO)))(K_0K_2\mu_1/T + K_0K_2\mu_3/T$$

$$+K_0K_3K_4\mu_4/T)$$

$$+\ 2(E(\widetilde{STK}(\tilde{\gamma},\ EBO)X_3(T)) - p_3\lambda T\ E(\widetilde{STK}(\tilde{\gamma},\ EBO)))(K_0K_1/T + K_0K_2\mu_1/T$$

$$+K_0K_2\mu_3/T + K_0K_3K_5\mu_5/T)$$

$$= K_0^2\ \mathrm{Var}(\widetilde{STK}(\tilde{\gamma},\ EBO)) + \mathrm{Var}(MLSC')$$

$$+\ (2/T)(K_0K_2\mu_1 + K_0K_2\mu_3 + K_0K_3K_4\mu_4)\mathrm{Cov}(\widetilde{STK}(\tilde{\gamma},\ EBO),\ X_2(T))$$

$$+\ (2/T)(K_0K_1 + K_0K_2\mu_1 + K_0K_2\mu_3 + K_0K_3K_5\mu_5)\mathrm{Cov}(\widetilde{STK}(\tilde{\gamma},\ EBO),\ X_3(T)) \quad \text{Q.E.D.}$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
In recent years, several new contractual arrangements have been devised to estimate, target, and track logistic support costs during the acquisition phase. One of these is a contractual mechanism known as a Logistic Support Cost Commitment (LSCC), sometimes referred to as a Support Cost Guarantee. The objective of the LSCC is to motivate the contractor to design his equipment to have reduced logistic support costs through increased reliability and maintainability (R&M) when fielded. → over

20. Abstract (Continued)

This report documents research into the statistical properties of the LSCC. The LSCC utilizes one of a broad class of statistical estimators, which are complex mathematical functions of simpler estimators whose statistical properties are well known. In the LSCC case, the complex estimator is a cost function, and incorporates such simpler estimators as rates of occurence, durations of activity, and physical distribution of activity. It also includes constant cost rates.

The research documented is primarily mathematical. It does not treat in-depth the numerous qualitative issues regarding LSCC use. It is addressed to analysts in Operations Research and Statistics, to whom its potential usefulness goes far beyond the LSCC problem. Analogies can be drawn to numerous other problems which appear in different settings and are stated in different nomenclature but which are, in abstract form, the same.